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Integrating a Computer Algebra System as the Pedagogical Tool for Enhancing Mathematical Thinking in Learning Differential Equations

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ABSTRACT

This study reports the design of teaching experiments in cooperation with a computer algebra system as the pedagogical tool to develop mathematical thinking in differential equations. Teaching experiment was conducted as a methodology. Researchers made instruments based on mathematical thinking approach proposed by Mason and instrumental genesis. Participants were chosen from a public university in Malaysia. Data was collected in several intervention sessions. To create appropriate categories, data was coded and analyzed based on qualitative data analysis. Double coding procedure has been done to enhance the trustworthiness of the research. In using the computer algebra system as pedagogical tool in learning differential equation, students have successfully adopted mathematical powers such as specializing, generalizing, and convincing while problem solving. Appropriate tasks, the role of teacher as an orchestrator, and ample questions and prompts to explicate mathematical thinking shifts from a conventional teaching and learning environment to an integrated computer algebra system environment. The findings can be applied to create a framework to use technology in mathematics education at undergraduate level. The created instruments can be used by educators and lecturers to enhance the quality of teaching, learning and mathematical thinking powers at undergraduate level.

Keywords: *Integrating cognitive tools, Differential equations, computer algebra system, mathematical thinking.*

INTRODUCTION

A CAS offers some possibilities and may be considered an idiosyncratic tool by students rather than a flexible instrument with informal notations and strategies (Drijvers, 2000). The enhancement of conceptual mathematics understanding and technical understanding of the CAS may influence one another (Artigue, 2002; Drijvers et al., 2010; Guin & Trouche, 1998; Lagrange, 1999). However, having a deeper look into student's knowledge while they are trying to solve mathematics problems with the CAS is required.

There are rich bodies of research emphasizing the use of technology; however, the paucity of the research in post secondary mathematics teaching is clear (Bishop, Clements, Keitel, Kilpatrick, & Leung, 2003; Hoyles & Lagrange, 2009; Lavicza, 2010; Marshall, Buteau, Jarvis, & Lavicza, 2012). CASs and the other cognitive tools help to transcend the limitation of the students' thinking, learning, and problem solving abilities (Snyder, 2007). They additionally may act as an amplifier and reorganizer. The accessibility of CASs, specifically open sources, offers new possibilities for classroom activities. The activities performed in a CAS environment can be substantially modified from a traditional approach to one that is proposed in Table 1.

Table 1: Modification of traditional approach in a CAS environment (Snyder, 2007:18)

From	To
Doing	Planning, interpreting
Reproductive learning	Active, experimental learning
Teacher- oriented learning	Pupil centered learning
Using predefined strategies	Developing strategies
Knowledge about calculations	Knowledge about strategies
Complex calculation skills	Use of a CAS for complex calculations
Exercise	Problem solving (modeling and interpreting)
Calculation oriented learning	Application oriented learning

CASs can help to achieve educational benefits such as exploring mathematics, but there are more benefits that can be attained when students struggle toward instrumentation, which validates the appropriation of the CAS as a helpful tool (Artigue, 2002). However, while the issues CAS integration in the mathematics classroom have been widely reported, an even more critical issue described by Ruthven (2007) is that facilitating mathematics classrooms with computer labs causes higher demand on the activities required to control the classroom (Lavicza, 2007). This article intends to look at the impact of the proposed environment on students' mathematical thinking in differential equations using computer algebra system.

THEORETICAL BACKGROUND

Mathematical thinking

A helical picture of the framework originally proposed by Bruner *et al* (1986) has been extended by Mason to represent mathematical thinking as a dynamic process. It moves around or between a number of unspecified loops, where each new loop is built on the students' understanding and awareness perceived in traversing previous loops (Burton, 1984). Based on Mason's definition (2000) mathematical thinking is a dynamic process that enables people to increase the complexity of their ideas. Therefore, they can handle and expand their understanding. He believes that people use mathematical thinking in four distinct ways including: Specializing, which refers to trying special cases and looking at examples; Generalizing, or looking for patterns and relationships; Conjecturing, which is the ability to predict relationships and results; and Convincing, which is the ability to find and communicate the reasons why something is true.

Three worlds of mathematics

The recent work of Tall (2008) is about transition in thinking from school mathematics to formal proofs in pure mathematics at the university level, which is formulated as the framework of the three worlds of mathematics to explain the process of the students' construction of their schemas. The conceptual embodied world includes perception, action and thought, the proceptual symbolic world is calculations based on mathematical symbols, and the axiomatic-formal world includes mathematical abstraction and proofs.

Mathematics knowledge is the ability to respond to problematic situations by construction or reconstruction processes, and objects explain the main aspects of mathematical epistemology. Additionally, reconstruction time could be different due to the requirements of the particular situation. On one side, within APOS theory (Action, Process, Object, and Schema) teachers help learners to construct appropriate mathematical mental structures. Moreover, students are guided to use the structures to construct their conceptual understanding of mathematics. On the other side, learning is assisted if the student's mental structures are appropriate to a given mathematical concept. They discuss their results and listen to explanations, by fellow students or the teacher, of the mathematical meaning of what they are working on (Dubinsky & Mcdonald, 2002).

Instrumental genesis

Tool use does not happen in a vacuum as tools are applied in an act, practice or a context. However, how individuals look at activities and practices are very important. Computers based on elaborating the psychological idea of cognitive tools in education have the power to both amplify and reorganize mathematical thinking (Pea, 1987). Furthermore, according to Drijvers *et al* (2010) and based on instrumental genesis, the duality between instrumentation and instrumentalization comes down to the pupils' thinking and while it is being shaped by the artifact, it also shapes the instrument. As a whole, instrumentation refers to the instrumentation in instrumental approaches; as a specific context of instrumental approaches, it refers to the way in which the artifact influences the student's thinking and behavior as opposed to instrumentalization which emphasizes the way the students' thinking affects the artifact. Students can develop operational facilities through mastering and elaborating instrumented activity with critical components of the conceptual system (Kenneth Ruthven, 2002). Well designed tasks can help students to demonstrate their mathematical thinking as they can develop and interpret situations (Serrano, 2012).

METHOD

Teaching experiment (TE) is a primary reason to experience students' mathematical learning and reasoning first hand. Steffe & Thompson (2000) claimed there would be no basis to understand powerful mathematical operations and concepts that students construct without the experiences obtained from teaching experiment. A TE is an experimental tool methodology used to answer teaching-research questions, determine the nature of learning mathematics, the development of mathematical thinking of the students, the role of the interaction in the classroom and many other issues of interest to teachers, researchers, and teacher-researchers (Czarnocha & Maj, 2006). Figure 1 shows the elements of a teaching experiment.

Conjecture

According to Confrey & Lachance (2000), conjecture in a teaching experiment is a means to re-conceptualize the ways in which to approach both the content and the pedagogy of a set of mathematical topics. In total, the intervention required 16 sessions over 9 weeks in one academic semester to implement by one teacher and one researcher. Tasks and worksheet activities have been done for 30 minutes per intervention session, which includes the content that is supposed to be presented.

Curriculum

The curriculum used in the classroom determined based on conjecture and what students and teachers are supposed to cover over a period of time. Content includes ordinary differential equations which are presented in Malaysian Public universities. Maxima was chosen as a computer algebra system because it is free and open source and its language is very similar to language used in mathematics discourse.

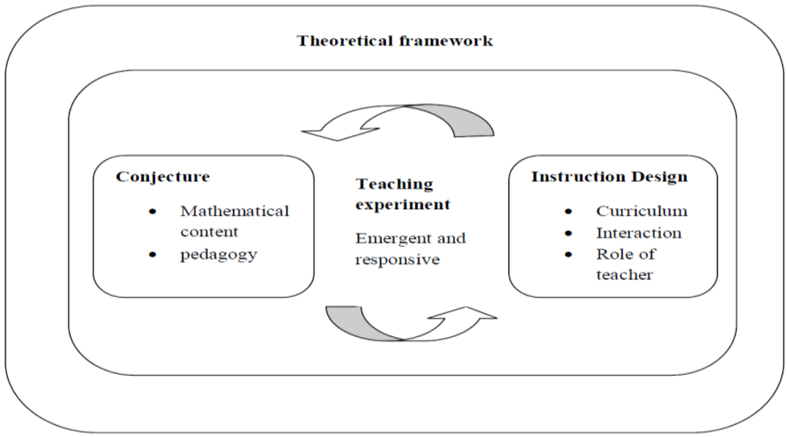


Figure 1: Teaching experiment design

Worksheets

The worksheets consist of two parts including written and computer lab activities. Both activities are covered in one syllabus in two ways. Questions and prompts created by Mason and Watson (1998) were modified in terms of differential equations and computer algebra system applications to draw students attention to mathematical processes and structures involved in mathematical thinking (Rahman, Yusof, & Baharun, 2012). Figure 2 shows structures in mathematics, and the order is not important. By considering mental activities and mathematical structures, a grid (see Figure 3) was produced to help generate particular questions for the topic under study.

Worksheets were created based on the hypothesis that using some CAS information capabilities such as high speed calculation and visualization using questions and prompts can enhance mathematical thinking powers such as specializing and generalizing, conjecturing, and convincing. Mason believes that core mathematical themes, powers, and activities are provoked explicitly; students are more likely to integrate them into their sense of mathematical thinking and into their own thinking than if they remain implicit or beneath the surface (Mason, 2000). Four central mathematical powers including specializing, generalizing, conjecturing, and convincing are considered in this research. However, these powers are not hierarchical. Three mathematical themes among many are explained including invariance amidst change, doing and undoing and freedom and constraint (Mason, 2000).

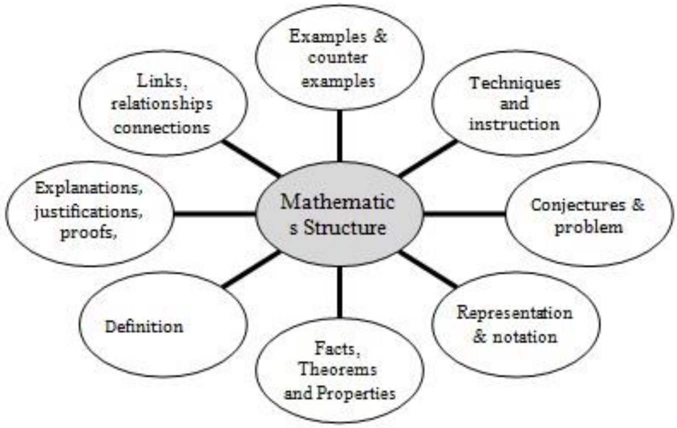


Figure 2: Mathematics structure

Invariance amidst change is considered a theme which is connected to a large number of theorems such as differentiability which is invariant under addition and multiplication by a scholar. Doing and undoing are considered a theme for fruitful and challenging questions. When you do a question and get an answer, it is useful to ask yourself, if someone gave you the answer, could you reconstruct the question. Reversing a question and its answer, inserting parameters and asking for conditions in which a certain problem is solvable is essential in problem solving (Mason, 2000).

Mathematics is a field in which the effects of constraints are analyzed and codified as techniques. Taking a problem to find and view it as a construction problem starting with a very general situation and then imposing constraints will make you aware of the features, characteristics and conditions of the various techniques. Therefore, freedom and constraint are considered important themes in mathematics. To understand a concept in mathematics, it is necessary to be aware of the features of a mathematical object that can vary freely, and the features which may or may not be constrained in some way. Within this frame, mathematics is considered as support for constructing the objects which meet constraints, rather than as a set of tools for producing answers on tests (Carlson & Rasmussen, 2008). Table 2 shows some typical questions related to mathematical themes.

	Exemplifying, specializing	*Completing, deleting and correcting*	*Comparing, sorting and organizing*	*Changing, varying, reversing and altering*	*Generalizing, conjecturing*	*Explaining, justifying, verifying convincing, refuting*
Definitions	**Show me an example of a**	**What must be added without affecting the equation?**	**What is the same and what different?**	**Change …so that it describes a…**	**Describe all … in one word, one sentence?**	**What is the definition of a DE?**
Facts, theorems and properties	**Tell me something that must be true if…**	**To be… complete and delete or correct properties as required?**	**What is the same and what is the different …?**	**What… do you get if you change …to…?**	**Is it always, sometimes, never, …**	**How can be sure that the solution given is true?**
Examples, counter-examples	**Which features of …make it an example of …**	**What (additional) properties must a … have so that it is an example of a**	**Sort the following according to the method used to solve**	**Change one aspect of the example so that…**	**Of what … an example?**	**Tell me what is wrong with…**
Techniques and instruction	**Tell me how to solve….**	**Complete the solution, Delete unnecessary steps**	**Is … a technique for….?**	**Show that every member of the family of functions ... is a solution of…**	**What can change, and what must still stay the same?**	**How can we be sure that …? Is it always true that…?**
Conjectures and problems	**Tell me what your current conjecture is.**	**What other information is needed in order to answer the question..?**	**What is the same and what is different about…?**	**How would your conjecture change if you changed…to…?**	**Describe all… in one word , one sentence, one diagram**	**Why do you want to solve a differential equation?**
Representation and notations	**Show me a way to (write, depict, graph, use calculator)**	**Complete the missing parts in the …**	**Is … a useful notation for…?**	**Write an equivalent first ODE.**	**Of what is…. An example?**	**Justify why….has slope fields as shown in the diagram?**
Explanations, Justification, proofs and reasoning	**How would you explain(justify)**	**Provide and insert missing steps? Correct the following steps?**	**What is the same and what is different about…?**	**Explain, justify, prove if we know …but do not know …**	**Describe possible solutions for the slope field.**	**Could you give a general rule to solve differential equations graphically?**
Links, relationships and connections	**Give an example of relationship between … and ….**	**…and … are the same in that both are … but different in that …**	**Make a connected chain from…**	**What if….?**	**Of what is…. An example?**	**Explain connection between graphical and analytical solutions of a differential equation.**

Figure 3: Questions and prompts to explicate mathematical thinking extended from (Watson & Mason, 1998)**.**

Figure 4 shows a typical worksheet used to integrate a computer algebra system with mathematical thinking approach. For example, item I-1-a is related to definition and specializing. I-1-b is related to technique structure and specializing powers that can be done through introducing symbols manually and II-1-b can be done by computer. What is generalized and articulated in a statement may lead to convincing powers using justification that students use to answer item I-1-d.

Table 2: Mathematical mapping questions according to themes

Themes	Questions
Invariance amidst change	What changes and what stays the same as you employ the technique of…..? What are the most important characteristics for general solutions? What is the same and what is the different of general solution and particular solution.
Doing and undoing	The solution of a differential equation is given and what differential equation the equation corresponds to solution?
Freedom and constraints	Find general solution of a differential equation with initial value y'(0)=1, and y(0)=1

Item II may lead to mathematical knowledge development based on instrumented technique development, instrumenting graphic and symbolic reasoning in CAS environment. Without laborious calculations, students can see the behavior of the solution of a given differential equation in item II-1 and II-3. Specializing powers include identifying facts, theorems and properties, and the techniques to help students solve items II-1-c and II-1-d. Damping situations related to second order differential equations with mathematical approach is shown in Figure 5. It is hypothesized that by using visualization and doing calculation students can distinguish three kinds of damping situations such as critical, over, and under damping

(I) Written activity:

I-1 Find the general and particular solution of given differential equation:

1. $y' = \frac{1}{x}$, $y(0) = 0$

I-1-a: Is it a linear differential equation? If so, why?

I-1-b: How do you find the solution of the equation?

I-1-c: What is the same and what is the different to solve this kind of equation and the others?

I-1-d: Does the equation have any solution at x=0? Justify your answers.

(II) Computer Lab activity

Use Maxima to answer the questions:

II-1 Plot the slope fields of the differential equation $y' = \frac{1}{x}$.

II-2 What is the particular solution for initial condition $y(0) = 0$?

II-1-a: What do you know about the command to use?

II-1-b: What do you want to do in Maxima environment?

II-1-c: How do you introduce y(0)=0 to Maxima?

II-1-d: What is the application of uniqueness theorem for solutions a differential equation here?

II-3 Is any solution curve pass through the point (0,0)? Verify your solution.

……………………………………………………………………………………………

II-4 What is your conclusion? ……………………………………………………………………

Figure 4: A typical mathematical thinking approach worksheet

Teaching experiment

Tasks and activities applied in the classroom form the classroom interactions during the intervention. These interactions are necessary in planning the data collection methods. In doing so, 37 students divided into 8 groups of four and one group of 5 during the intervention. Tasks and activities, which include the use of technology, form classroom interactions such as individual assignments, group assignments, exercises, activities involved in worksheets, and quizzes. These interactions consist of interaction with differential equation concepts, technology (Maxima), group members, the teacher, along with the screen on the board.

The role of the teacher in transformative and conjecture-driven teaching depends on who acts as a teacher during the intervention and what role the teacher plays during the intervention. Many research studies emphasize the role of teacher in Instrumental genesis. Thus, the teacher-researcher in the second part is the exemplary user of the tool who orchestrates the instrumentation of the CAS (Maxima) by means of individual interactions, discussions and demonstrations in differential equation classrooms (Trouche, 2004). The researcher- second author- plays a role as teacher- researcher to conduct the intervention sessions.

The outcome of assessments not only provides data on the impact of the intervention, but also helps in the ongoing formative evaluation of the intervention process. These quizzes conducted in the class to gain information about development in the students understanding of differential equations.

(I) Written activity:

I-Find the general solutions of the following equations.

I-1: $y'' + y' + 3y = 0$

I-2: $y'' + 4y' + 3y = 0$

I-3: $y'' + 4y' + 4y = 0$

I-1-a: Are the equations linear or non-linear?
I-1-b: what are the roots of auxiliary equations?
I-1-c: Which one is under-damping? Why? Justify your solution
I-1-d: Which one is critical-damping? Why? Justify your solution
I-1-e: Which one is over-damping? Why? Justify your solution

(II) Computer Lab activity

II-1: Draw the graph of the solutions the given differential equations in written activity (I) using Maxima in which $y(0) = 1,\ y'(0) = 0$?

II-2: Using the graph, show the under, over and critical damping?

II-3: How are you sure your answers to item II-2 are correct?

....

II-1-a: What do you know about the command to get the solutions of item II-1?
II-1-b: How do you introduce the $y(0) = 1,\ y'(0) = 0$ to Maxima?
II-1-c: Describe the damping situation that the graph describes?
II-1-d: What is the same and what is the difference between the differential equations and the graphs of them that caused you recognize the damping situation?

Figure 5: A typical worksheet with mathematical thinking approach in second order differential equations

The TE was organized during semester I 2012/2013 in a 37 student differential equation classroom at a public university in Malaysia. 6 participants consisting of higher, middle, and lower achievers were chosen to be interviewed in depth to investigate the phenomenon under study. A semi-structured interview with open-ended questions and a task based interview conducted based on the protocol interview after completing all intervention sessions to identify the student development. In addition, data collected through observation notes, student' interactions in the classroom, student written activities, and researchers' thinking notes.

The first kind of data analysis is called ongoing or preliminary analysis which occurs during the intervention. The second type of analysis occurred after the classroom activities have been finished. The methods of data analysis were based on the methods of data collection and the results of preliminary analysis, and the conjecture. The process of coding and creating a category system for analyzing qualitative data were used among other analytic techniques (Patton, 2001). Double coding procedure was conducted to promote the trustworthiness of the research through a panel including the researchers who applied the same framework in their research and have published in both differential equation and mathematical thinking areas.

FINDINGS

Findings showed students use symbols as a specializing power. However, in CAS environment this power can be supported by the removal of tedious calculation. To do the activities shown in Figure 5, students used the same command to obtain several different of damping situations (line 2,6,9,15,16,17,19,20,21). Some students found the specific example in the help sheet, then copied and modified the commands based on the given differential equation to solve. However, some participants prefer to write the appropriate commands to give the solution.

1 *Researcher: could you please solve this equation using Maxima?*

$$y'' + y' + 3y = 0$$

2 *Student: eqn:'diff(y,x,2)+'diff(y,x)+3*y=0; to introducing an equation for Maxima.*

(%o1) $\frac{d^2}{d x^2} y + \frac{d}{d x} y + 3\, y = 0$

3 *Researcher: what is the order of the differential equations?*

4 *Student: two, the highest is two*

5 *Researcher: how do you understand?*

6 *Student: because of diff(y,x,2)*

7 *Student: gsoln:ode2(eqn,y,x); for getting the general solution*

$$y=\%e^{-\frac{x}{2}}\left(\%k1\,\sin\left(\frac{\sqrt{11}\,x}{2}\right)+\%k2\,\cos\left(\frac{\sqrt{11}\,x}{2}\right)\right)$$

8 *Student: Can I check the particular solution syntax in the help sheet.*

9 *psoln:ic2(gsoln,x=0,y=1,'diff(y,x)=0);*

$$y=\%e^{-\frac{x}{2}}\left(\frac{\sin\left(\frac{\sqrt{11}\,x}{2}\right)}{\sqrt{11}}+\cos\left(\frac{\sqrt{11}\,x}{2}\right)\right)$$

10 *Student: Now I got the solution.*

11 *Researcher: what is the difference between anything you got manually and with computer?*

12 *Student: Nothing, both are the same*

When students are given two other kinds of differential equations, they simply use the same command to find the solutions without any laborious calculations. It can help students to conjecture about damping situations based on the changes in the algebraic and graphical solutions. The following example shows the students experimentation in critical, over, and under damping.

13 *Researcher: could you please solve these two more de*

I-2: $y''+4y'+3y=0$

I-3: $y''+4y'+4y=0$

14 *Student: for solving this* $y''+4y'+3y=0$

15 *ode:'diff(y,x,2)+4*'diff(y,x)+3*y=0;*

$$\frac{d^2}{d\,x^2}\,y+4\left(\frac{d}{d\,x}\,y\right)+3\,y=0$$

16 *gsoln:ode2(ode,y,x);*

$$y=\%k1\,\%e^{-x}+\%k2\,\%e^{-3\,x}$$

17 *psoln:ic2(gsoln,x=0,y=1,'diff(y,x)=0);*

$$y=\frac{3\,\%e^{-x}}{2}-\frac{\%e^{-3\,x}}{2}$$

18 *Student: for solving* $y''+4y'+4y=0$

19 *ode:'diff(y,x,2)+'diff(y,x)+3*y=0;*

$$\frac{d^2}{d\,x^2}\,y+\frac{d}{d\,x}\,y+3\,y=0$$

20 *gsoln:ode2(ode,y,x);*

$$y=\%e^{-\frac{x}{2}}\left(\%k1\,\sin\left(\frac{\sqrt{11}\,x}{2}\right)+\%k2\,\cos\left(\frac{\sqrt{11}\,x}{2}\right)\right)$$

21 *psoln:ic2(gsoln,x=0,y=1,'diff(y,x)=0);*

$$y=\%e^{-\frac{x}{2}}\left(\frac{\sin\left(\frac{\sqrt{11}\,x}{2}\right)}{\sqrt{11}}+\cos\left(\frac{\sqrt{11}\,x}{2}\right)\right)$$

Visualization in Maxima was appreciated by almost all participants. Using a command such as *wxplot2d* they could get the graph of the solution of a given differential equation. Line 23 shows the use of the one command to see all three damping situations together.

22 *Researcher: could you please distinguish which kinds of damping they are?*

23 *Student: let me draw first: wxplot2d([Graph1,Graph2,Graph3],[x,0,2])*

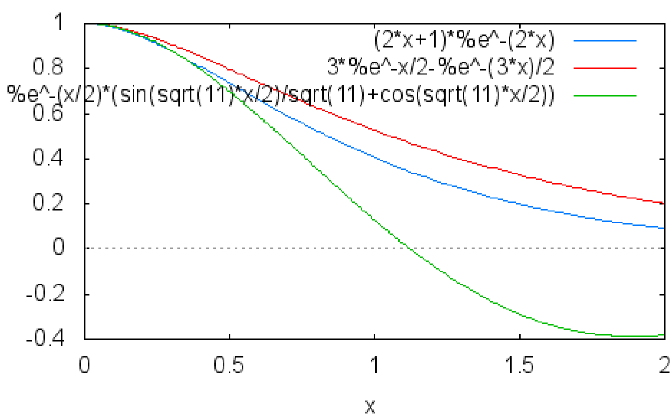


Figure 6: Three damping systems in one graph.

Higher achievers put all three damped situations in one graph (Figure 6). However, participants explored the situation individually through Maxima using three different graphs without demonstrating difficulties. The graphs helped students to see the relationship between symbolic and graphical solutions. Question and prompts like line 24 are used to reinforce generalizing powers such as checking the calculation to ensure generalization is true. Moreover, they checked the argument to ensure that the computations are appropriate.

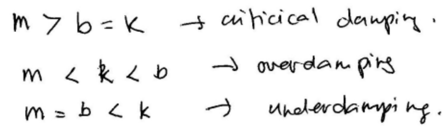
24 *Researcher: what is the same and what is the difference between the symbolic solutions and graphical solution that you got?*

25 *Student: In over damping red one, the graph is suddenly fall to zero, I mean approach to zero. I can see for =(3*%e $^{(-x))/2}$-%e $^{(-3*x)/2.}$ However, in green one, this is a trigonometric function, waving, so this graph is correct*

In relation to making conjectures students were asked to articulate their guess into a statement and to write it on the piece of paper (line 27). An example is below:

26 *Researcher: could you please what is the same and what is difference between the differential equations of these three kinds of graph and damping situations? Could you please write down your conjecture as a statement?*

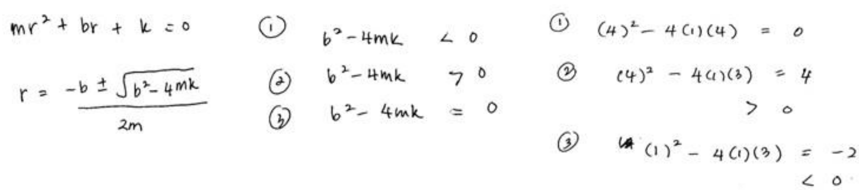
27 *Student:*



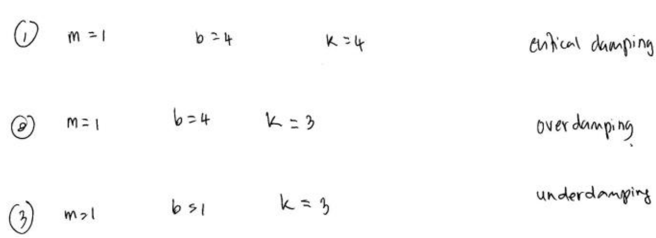
Having written a statement related to their conjectures, students were prompted to convince the researcher that their conjectures were correct. An example in line 29 shows that students tried to justify the correctness of the conjecture already made. The justification is shown in line 30.

28 Researcher: How are you sure your statement is correct? Could you please justify it?

29 Student:



30 Therefore it based on the damping that I got in Maxim symbolically and graphically:



DISCISSION AND CONCLUSION

This study was conducted to integrate cognitive tools such as a computer algebra system into differential equations to promote mathematical thinking powers. Several worksheets were made based on questions and prompts by Mason created by researchers to enhance mathematical thinking. Moreover, instrumental genesis and the use of Three worlds of mathematics as the theoretical framework strengthen the findings of the research. Data was collected from approximately 17 intervention sessions in differential equation classrooms and several interviews.

Findings showed students' mathematical thinking powers can be stimulated if they are prompted appropriately by a computer algebra system. Cognitive tools can manipulate high speed calculation and graphs. The findings confirm Artigue (2002), which expresses two major ways in which using CAS can lead to mathematical knowledge development including instrumented technique development, facilitating and extending experimentation with mathematical systems such as generalization. In addition, the results confirm (2002) that a good design with a specific objective can help students to demonstrate their mathematical thinking (Serrano, 2012). Moreover, according to Ruthven (2002) students can develop operational facilities through mastering and elaborating instrumented activity with critical
components of the conceptual system. Due to limitation of words, only a small part of the results are presented in the article.

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