Curvature inequalities for $C$–totally real doubly warped products of locally conformal almost cosymplectic manifolds

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Abstract. In this paper, we establish some optimal inequalities for the squared mean curvature in terms warping functions of a $C$–totally real doubly warped product submanifold of a locally conformal almost cosymplectic manifold with a pointwise $\varphi$–sectional curvature $c$. The equality case in the statement of inequalities is also considered. Moreover, some applications of obtained results are derived..

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1. Introduction

The idea of warped product manifolds was first introduced by Bishop and O'Neill (cf. [5]) to study manifolds of negative curvature. Later on, doubly warped product manifolds studied by Unal [29]. They defined these manifolds as follows:

Definition 1.1. Let $M_1$ and $M_2$ be two Riemannian manifolds of dimensions $n_1$ and $n_2$ endowed with Riemannian metrics $g_1$ and $g_2$ such that $f_1 : M_1 \to (0, \infty)$ and $f_2 : M_2 \to (0, \infty)$ be positive differentiable functions on $M_1$ and $M_2$, respectively. Thus, the doubly warped product $M = f_2 \times f_1 M_1 \times M_2$ is defined to be the product manifold $M_1 \times M_2$ with equipped metric $g = f_2^2 g_1 + f_1^2 g_2$. Moreover, if we consider $\gamma_1 : M_1 \times M_2 \to M_1$ and $\gamma_2 : M_1 \times M_2 \to M_2$ are the natural projections on $M_1$ and $M_2$, respectively then the metric $g$ on doubly warped product is defined as

$$g(X, Y) = (f_2 \circ \gamma_2)^2 g_1(\gamma_1 X, \gamma_1 Y) + (f_1 \circ \gamma_1)^2 g_2(\gamma_2 X, \gamma_2 Y) \quad (1.1)$$

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for any vector fields $X, Y$ tangent to $M$, where $\star$ is the symbol for the tangent maps. Thus, the functions $f_1$ and $f_2$ are called warping functions on $M$. If both $f_1 = 1$ and $f_2 = 1$, then $M$ is called a simply Riemannian product manifold. If either $f_1 = 1$ or $f_2 = 1$, then $M$ is called a (single) warped product manifold. If $f_1 \neq 1$ and $f_2 \neq 1$, then $M$ is called a non-trivial doubly warped product manifold. Let $M = f_1 M_1 \times f_2 M_2$ be a non-trivial doubly warped product manifold of an arbitrary Riemannian manifold $M$. Then

$$
\nabla_X Z = \nabla_Z X = (\ln f_2)X + (\ln f_1)Z,
$$

$$
\nabla_X Y = \nabla_X^1 Y - \frac{f_2^2}{f_1^2} g_1(X, Y)(\ln f_2),
$$

for any vector fields $X, Y \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$. Further, $\nabla^1$ and $\nabla^2$ are Levi-Civita connections of the induced metrics on Riemannian manifolds $M_1$ and $M_2$, respectively.

The following well-known result of Chen in [8] obtained a sharp relationship between the squared norm of mean curvature and the warping function $f$ of warped product $M \times_f M$ isometrically immersed into a real space form.

**Theorem 1.2.** [8]. Let $x : M_1 \times_f M_2$ be an isometrically immersion of an $n$-dimensional warped product into a $2m$-dimensional real space form $\tilde{M}(c)$ with constant sectional curvature $c$. Then

$$
\frac{\Delta f}{f} \leq \frac{n_1^2}{4n_2} ||H||^2 + n_1 c
$$

where $n_i = \dim M_i$, $i = 1, 2$ and $\Delta$ is the Laplacian operator of $M_1$. Moreover, the equality holds in the above inequality if and only if $x$ is mixed totally geodesic and $n_1 H_1 = n_2 H_2$ such that $H_1$ and $H_2$ are partial mean curvature.

Motivated by Chen’s result several inequalities have been obtained by many geometers for warped products and doubly warped products in different setting of the ambient manifolds [9, 18, 24, 25, 30, 34, 35]. In this paper, we study to $C$–totally real doubly warped product isometrically immersed into a locally conformal almost cosymplectic manifold. The inequalities which we obtain in this paper are very fascinating because we derive upper bound and lower bound for warping functions in terms of mean curvature, scalar curvature and pointwise constant $\varphi$–sectional curvature $c$. The obtained results generalise some other inequalities as special cases.

**2. Preliminaries**

A $(2m + 1)$-dimensional smooth manifold $\tilde{M}$ is called locally conformal almost cosymplectic manifold, if it is consisting an endomorphism $\varphi$ of its tangent bundle $T\tilde{M}$, a structure vector field $\xi$ and a 1-form $\eta$ which satisfies the following:

$$
\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \varphi = 0,
$$

$$
g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad \eta(U) = g(U, \xi),
$$

$$
(\nabla_U \varphi)V = \delta [g(\varphi U, V) - \eta(V)\varphi U],
$$

$$
\nabla_U \xi = \delta [U - \eta(U)\xi],
$$

for any $U, V$ tangent to $\tilde{M}$ and $\omega = \delta \eta$ (see [26]). Let us we consider that the function $\vartheta = 0$ and $\vartheta = 1$, then $\tilde{M}$ becomes cosymplectic manifold and Kenmotsu manifold, respectively (see [14, 34]). An almost contact metric manifold $\tilde{M}$, a plane section $\sigma$ in $T_p\tilde{M}$ of $\tilde{M}$ is said to be a $\varphi$–section if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. The sectional curvature $\tilde{K}(\sigma)$ does not depend on the choice of the $\varphi$–section $\sigma$ of $T_p\tilde{M}$ at each point $p \in \tilde{M}$, then $\tilde{M}$ is called a manifold with pointwise constant $\varphi$–sectional curvature. In this case for any $p \in \tilde{M}$ and for $\varphi$–section $\sigma$ of $T_p\tilde{M}$, the function $c$ defined by $c(p) = \tilde{K}(p)$ is said to be $\varphi$–sectional curvature of $\tilde{M}$. That is, for
a locally conformal almost cosymplectic manifold \( \tilde{M} \) of dimension \( \geq 5 \) with pointwise \( \varphi \)-sectional curvature \( c \), its curvature tensor \( \tilde{R} \) is defined as

\[
\tilde{R}(X, Y, Z, W) = \frac{c - 3\delta^2}{4} \left[ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right] \\
+ \frac{c + \delta^2}{4} \left[ g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W) \right] \\
- 2g(X, \varphi Y)g(Z, \varphi W) \\
- \left[ \frac{c + \delta^2}{4} + \delta' \right] \left[ g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W) \\
+ \eta(Y)\eta(W)\eta(X)\eta(Z) \right],
\]

(2.5)

for any \( X, Y, Z, W \) are tangent to \( \tilde{M}(c) \), where \( \delta \) is the conformal function such that \( \omega = \delta \eta \) and \( \delta' = \xi \delta \). Moreover, \( c \) is the function of constant \( \varphi \)-sectional curvature of \( \tilde{M} \). If we consider the function \( \delta = 0 \) and \( \delta = 1 \), then \( \tilde{M}(c) \) becomes cosymplectic space form and Kenmotsu space form, respectively (see [18, 34]). Let us consider that \( M \) be a submanifold of an almost contact metric manifold \( \tilde{M} \) with induced metric \( g \) and if \( \nabla \) and \( \nabla^\perp \) are the induced connections on the tangent bundle \( TM \) and the normal bundle \( T^\perp M \) of \( M \), respectively, then Gauss and Weingarten formulas are given by

\[
(i) \, \tilde{\nabla}_U V = \nabla_U V + h(U, V), \quad (ii) \, \tilde{\nabla}_U N = -A_N U + \nabla^\perp_U N,
\]

(2.6)

for each \( U, V \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \), where \( h \) and \( A_N \) are the second fundamental form and the shape operator (corresponding to the normal vector field \( N \)) respectively for the immersion of \( M \) into \( \tilde{M} \). They are related as

\[
g(h(U, V), N) = g(A_N U, V),
\]

(2.7)

where \( g \) denotes the Riemannian metric on \( \tilde{M} \) as well as the metric induced on \( M \). Now, for any \( U \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \), we have

\[
(i) \, \varphi U = TU + FU, \quad (ii) \, \varphi N = tN + fN,
\]

(2.8)

where \( TU \) (\( tN \)) and \( FU \) (\( fN \)) are tangential and normal components of \( \varphi U \) (\( \varphi N \)), respectively. From (2.1) and (2.5 (i), it is easy to observe that for each \( U, V \in \Gamma(TM) \), we have

\[
(i) \, g(TU, V) = -g(U, TV) \quad (ii) \, ||T||^2 = \sum_{i,j=1}^n g^2(Te_i, e_j).
\]

(2.9)

For a submanifold \( M \), the Gauss equation is:

\[
\bar{R}(U, V, Z, W) = R(U, V, Z, W) + g(h(U, L), h(V, W)) \\
- g(h(U, W), h(V, Z)),
\]

(2.10)

for any \( U, V, Z, W \in \Gamma(TM) \), where \( \bar{R} \) and \( R \) are the curvature tensors on \( \tilde{M} \) and \( M \), respectively. The mean curvature vector \( H \) for an orthonormal frame \( \{e_1, \cdots, e_n\} \) of tangent space \( TM \) on \( M \) is defined by

\[
H = \frac{1}{n} \text{trace}(h) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i),
\]

(2.11)

where \( n = \text{dim} \, M \). Also, for any \( r \in \{e_{n+1}, \cdots, e_{2m+1}\} \), we set

\[
h'_{ij} = g(h(e_i, e_j), e_r) \quad \text{and} \quad ||h||^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)).
\]

(2.12)
The scalar curvature $\rho$ for a submanifold $M$ of an almost contact manifold $\tilde{M}$ is given by

$$\rho = \sum_{1 \leq i,j \leq n} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the sectional curvature of plane section spanned by $e_i$ and $e_j$. Let $G_r$ be a $r$-plane section on $TM$ and $\{e_1, e_2, \cdots, e_l\}$ any orthonormal basis of $G_r$. Then the scalar curvature $\rho(G_r)$ of $G_r$ is given by

$$\rho(G_r) = \sum_{1 \leq i,j \leq l} K(e_i \wedge e_j).$$

A submanifold $M$ of an almost contact metric manifold $\tilde{M}$ is said to be totally umbilical and totally geodesic if $h(U, V) = g(U, V)H$ and $h(U, V) = 0$, respectively, for any $U, V \in \Gamma(TM)$, where $H$ is the mean curvature vector of $M$. Furthermore, if $H = 0$, then $M$ is minimal in $\tilde{M}$. On the other hand, a submanifold $M$ is called totally real submanifold if $T$ is identically zero, i.e. $\varphi U \in \Gamma(T^\perp)$ for any $U \in \Gamma(TM)$ for each $p \in M$.

Moreover, if the structure vector field $\xi$ is normal to submanifold $M$, then $M$ is said to be a $C$–totally real submanifold [19] of an almost contact manifold.

Let $\phi : M = f_\circ M_1 \times f_\circ M_2 \to \tilde{M}$ be isometric immersion of a doubly warped product $f_\circ M_1 \times f_\circ M_2$ into a Riemannian manifold of $\tilde{M}$ of constant sectional curvature $c$. Suppose that $n_1$, $n_2$ and $n$ are the dimensions of $M_1$, $M_2$ and $M_1 \times f_\circ M_2$, respectively. Then for unit vector fields $X$ and $Z$ tangent to $M_1$ and $M_2$ respectively, we have

$$K(X \wedge Z) = g(V_Z \nabla_X X - V_X \nabla_Z X, Z) = \frac{1}{f_1}((V^\perp_X f_1 - X^2 f_1) + \frac{1}{f_2}((V^\perp_X f_2 - Z^2 f_2).$$

(2.15)

If we consider the local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ such that $e_1, e_2, \cdots, e_{n_1}$ tangent to $M_1$ and $e_{n_1+1}, \cdots, e_n$ are tangent to $M_2$, then the sectional curvature in terms of general doubly warped product is defined by

$$\sum_{1 \leq i \leq n_1, n_1+1 \leq j \leq n} K(e_i \wedge e_j) = \frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2},$$

(2.16)

for each $j = n_1 + 1, \cdots, n$. Now, we have the following useful lemma.

**Lemma 2.1.** [8]. Let $a_0, a_1, \cdots, a_{n}, a_{n+1}$ be $n+1$ ($n \geq 2$) be real numbers such that

$$\sum_{i=1}^{n} a_i^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + a_{n+1}).$$

(2.17)

Then $2a_1a_2 \geq a_3$ with the equality holds if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

On the other hand, we analyze general doubly warped products in locally conformal almost symplectic manifold. That is, let $M = f_1, M_1 \times f_1, M_2 \to \tilde{M}$ be an isometric immersion from a doubly warped product $f_1, M_1 \times f_1, M_2$ into a locally conformal almost symplectic manifold $\tilde{M}$. Assume that $\xi \in \Gamma(TM_1)$ and $X \in \Gamma(TM_2)$, thus, from (2.4), we obtain

$$\vartheta h(X, \xi) = 0.$$

which implies by using (2.6) (i) and $\eta(X) = 0$, that

$$\nabla_X \xi \equiv \eta(X), \quad h(X, \xi) = 0.$$
Using (1.3) in the first relation of above equation, we find
\[(X \ln f_2)\xi + (\xi \ln f_1)X = \delta X.\]  
(2.18)

Now, taking the inner product with \(\xi\) in (2.18), we obtain \(X \ln f_2 = 0\), i.e., \(f_2\) is constant on \(M_2\). Hence, there is no doubly warped product in a locally conformal almost cosymplectic manifold, if \(\xi\) is tangent to \(M_1\). Moreover, if \(\xi \in \Gamma(TM_2)\) and \(Z \in \Gamma(TM_1)\), then again from (2.4), we have
\[\bar{\nabla}_Z \xi = \delta Z.\]

From (2.6) (i), we get
\[V_Z \xi = \delta Z, \quad h(Z, \xi) = 0.\]  
(2.19)

Again, using (1.3) in (2.19) and then taking the inner product with \(\xi\), it is easy to see \(f_1\) is also constant function on \(M_1\). Hence, in both the cases, any one of the warping function is constant. Thus, we conclude that there do not exist doubly warped product submanifold in a locally conformal almost cosymplectic manifold such that \(\xi\) is tangent to the submanifold. Therefore, we consider \(\xi\) is normal to submanifold \(M\) and there is a non-trivial doubly warped product in a locally conformal almost cosymplectic manifold which is called \(C\)–totally real doubly warped product. In the next section, we obtain some geometric inequalities for such type doubly warped product immersions.

3. Main inequalities of \(C\)–totally real doubly warped products

**Theorem 3.1.** Let \(\bar{M}(c)\) be a \((2m + 1)\)–dimensional locally conformal almost cosymplectic manifold and \(\phi : f_i M_1 \times f_j M_2 \rightarrow \bar{M}(c)\) be an isometric immersion of an \(n\)–dimensional \(C\)–totally real doubly warped product into \(\bar{M}(c)\) such that \(c\) is pointwise constant \(\varphi\)–sectional curvature. Then

(i) The relation between warping functions and the squared norm of mean curvature is given by
\[\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq \frac{n^2}{4} ||H||^2 + \frac{c - 3 \delta^2}{4} n_1 n_2,\]  
(3.1)

where \(n_i = \dim M_i, \quad i = 1, 2\) and \(\Delta^i\) is the Laplacian operator on \(M_i, \quad i = 1, 2\).

(ii) The equality sign holds in the above inequality if and only if \(\phi\) is mixed totally geodesic immersion and \(n_1 H_1 = n_2 H_2\), where \(H_1\) and \(H_2\) are the partial mean curvature vectors on \(M_1\) and \(M_2\), respectively.

**Proof.** Suppose that \(f_i M_1 \times f_j M_2\) be a \(C\)–totally real doubly warped product submanifold in a locally conformal almost cosymplectic manifold \(\bar{M}(c)\) with pointwise constant \(\varphi\)–sectional curvature \(c\). Then from Gauss equation (2.10) and (2.5), we derive
\[2\rho = \frac{c - 3 \delta^2}{4} n(n - 1) + n^2 ||H||^2 - ||h||^2.\]  
(3.2)

Let us consider that
\[\delta = 2\rho - \frac{c - 3 \delta^2}{4} n(n - 1) - \frac{n^2}{2} ||H||^2.\]  
(3.3)

Then from (3.2) and (3.3), it follows that
\[n^2 ||H||^2 = 2(\delta - ||h||^2).\]  
(3.4)

Thus from the orthonormal frame field \(\{e_1, e_2, \ldots, e_n\}\), the above equation takes the form
\[\left(\sum_{r=n+1}^{2m+1} \sum_{i=1}^{n} h_{ri}^2\right)^2 = 2(\delta + \sum_{r=n+1}^{2m+1} \sum_{i=1}^{n} (h_{ri})^2 + \sum_{r=n+1}^{2m+1} \sum_{i<j=1}^{n} (h_{ri})^2 + \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} (h_{ri})^2),\]
which simplifies as
\[
\frac{1}{2}\left(n_1^{n+1} + \sum_{i=2}^{n} h_{ii}^{n+1} + \sum_{i=n+1}^{n} h_{ii}^{n+1}\right) = \delta + (h_{11}^{n+1})^2 + \sum_{i=2}^{n} (h_{ii}^{n+1})^2 + \sum_{i=n+1}^{n} (\delta_{ii}^{n+1})^2 \\
- \sum_{2 \leq j \neq i \leq n_{1}} h_{jj}^{n+1} h_{ij}^{n+1} - \sum_{N_1+1 \leq i \leq n} h_{ii}^{n+1} \delta_{ii}^{n+1} \\
+ \sum_{i=1}^{n} (h_{ij}^{n+1})^2 + \sum_{r=m+1}^{2n} \sum_{i=1}^{n} (h_{ij}^r)^2.
\]
(3.5)

Assume that \(a_1 = h_{11}^{n+1}, a_2 = \sum_{i=2}^{n} h_{ii}^{n+1}\) and \(a_3 = \sum_{i=n+1}^{n} h_{ii}^{n+1}\). Then applying the Lemma 2 in (3.5), it is easily seen that
\[
\frac{\delta}{2} + \sum_{i=1}^{n} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=m+1}^{2m+1} \sum_{i=1}^{n} (h_{ij}^r)^2 \leq \sum_{2 \leq j \neq i \leq n_{1}} h_{jj}^{n+1} h_{ij}^{n+1} + \sum_{N_1+1 \leq i \leq n} h_{ii}^{n+1} \delta_{ii}^{n+1}.
\]
(3.6)
The equality holds in (3.6) if and only if
\[
\sum_{i=1}^{n} h_{ii}^{n+1} = \sum_{i=n+1}^{n} h_{ii}^{n+1}.
\]
(3.7)

On the other hand, from (2.13) and (2.16), we find that
\[
\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} = \rho - \sum_{1 \leq k \leq n_1} K(e_i \wedge e_i) - \sum_{n_1+1 \leq k \leq n} K(e_i \wedge e_i).
\]
From (2.5) and (2.10), it follows that
\[
\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} = \rho - \frac{c - 3\delta^2}{8} n_1(n_1 - 1) - \sum_{2 \leq j \neq k \leq n_1} (h_{ij}^r h_{kk}^r - (h_{ij}^r)^2) \\
- \frac{c - 3\delta^2}{8} n_2(n_2 - 1) - \sum_{1 \leq j \neq k \leq n} (h_{ij}^r h_{kk}^r - (h_{ij}^r)^2).
\]
(3.8)
Thus combining (3.6) and (3.8), it is easily seen that
\[
\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq \rho - \frac{c - 3\delta^2}{8} n(n - 1) + \frac{c - 3\delta^2}{4} - n_1 n_2 - \frac{\delta}{2}.
\]
(3.9)

Hence, from (3.3), the inequalities (3.9) reduce to
\[
\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq \frac{n^2}{4} ||H||^2 + \frac{c - 3\delta^2}{4} n_1 n_2
\]
(3.10)
which is the inequality (3.1). On the other hand, the equality sign holds in (3.10) if and only if from (3.7), we get \(n_1 H_1 = n_2 H_2\). Moreover, from (3.6), we find that \(h_{ij}^r = 0\), for each \(1 \leq i \leq n_1, n_1 + 1 \leq j \leq n\) and \(n_1 + 1 \leq r \leq 2m + 1\), which means that \(\phi\) is a mixed totally geodesic immersion. The converse part is straightforward. Thus, the proof is complete. \(\Box\)

Now, we have the following applications of Theorem 3.1
Remark 3.1. If we substitute either $f_1 = 1$ or $f_2 = 1$ in Theorem 3.1, then Theorem 3 turns into $C$–totally real warped product.

Corollary 3.1. Let $\tilde{M}(c)$ be a $(2m + 1)$–dimensional locally conformal almost cosymplectic manifold and $\phi : M_1 \times_f M_2 \to \tilde{M}(c)$ be an isometric immersion of an $n$–dimnesional $C$–totally real warped product into $\tilde{M}(c)$ such that $c$ is pointwise constant $\varphi$–sectional curvature. Then

(i) The relation between warping function and the squared norm of mean curvature is given by

$$\frac{n_2 \Delta f}{f} \leq \frac{n_2}{4} ||H||^2 + \frac{c - 3 \vartheta^2}{4} n_1 n_2,$$

where $n_i = \dim M_i$, $i = 1, 2$ and $\Delta$ is the Laplacian operator on $M_1$.

(ii) The equality sign holds in the above inequality if and only if $\phi$ is mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where $H_1$ and $H_2$ are the partial mean curvature vectors on $M_1$ and $M_2$, respectively.

Remark 3.2. If we put either $f_1 = 1$ or $f_2 = 1$ and $\vartheta = 0$ in Theorem 3.1, then it is the same inequality of Theorem 3.2 in [34].

Remark 3.3. If we consider either $f_1 = 1$ or $f_2 = 1$ and $\vartheta = 1$ in Theorem 3.1, then the Theorem 3.1 is exactly the Lemma 3.1 of [18].

Corollary 3.2. Let $\phi : M = f \phi M_1 \times_f M_2 \to \tilde{M}(c)$ be an isometric immersion of an $n$–dimnesional $C$–totally real doubly warped product into a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ with $c$, a pointwise constant $\varphi$–sectional curvature such that the warping functions are harmonics. Then, $M$ is not a minimal submanifold of $\tilde{M}$ with inequality

$$\vartheta > \frac{c}{\sqrt{3}}.$$

Corollary 3.3. Let $\phi : M = f \phi M_1 \times_f M_2 \to \tilde{M}(c)$ be an isometric immersion of an $n$–dimnesional $C$–totally real doubly warped product into a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ with $c$, a pointwise constant $\varphi$–sectional curvature. Suppose that the warping functions $f_1$ and $f_2$ of $M$ are eigenfunctions of Laplacian on $M_1$ and $M_2$ with corresponding eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Then $M$ is not a minimal submanifold of $\tilde{M}$ with inequality

$$\vartheta \geq \frac{c}{\sqrt{3}}.$$

Corollary 3.4. Let $\phi : M = f \phi M_1 \times_f M_2 \to \tilde{M}(c)$ be an isometric immersion of an $n$–dimnesional $C$–totally real doubly warped product into a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ with $c$, a pointwise constant $\varphi$–sectional curvature. Suppose that one of the warping function is harmonic and other one is eigenfunction of Laplacian with corresponding eigenvalue $\lambda > 0$. Then $M$ is not minimal in $\tilde{M}$ with inequality

$$\vartheta \geq \frac{c}{\sqrt{3}}.$$

Now, motivated by the Chen’s paper [9], we establish the following sharp relationship for the squared norm of the mean curvature vector in terms of intrinsic invariants.

Theorem 3.2. Let $\tilde{M}(c)$ be a $(2m + 1)$–dimensional locally conformal almost cosymplectic manifold and $\phi : M = f \phi M_1 \times_f M_2 \to \tilde{M}(c)$ be an isometric immersion of an $n$–dimnesional $C$–totally real doubly warped product into $\tilde{M}(c)$ such that $c$ is pointwise constant $\varphi$–sectional curvature. Then
\((\Delta_1 f_1)_{n_1 f_1} + (\Delta_2 f_2)_{n_2 f_2} \geq \rho - \frac{n^2(n-2)}{2(n-1)} ||H||^2 - \left(\frac{c - 3 \delta^2}{4}\right)(n + 1)(n - 2), \tag{3.11}\)

where \(n_i = \text{dim } M_i, \ i = 1, 2 \) and \(\Delta^i \) is the Laplacian operator on \(M_i, \ i = 1, 2.\)

(ii) If the equality sign holds in (3.11), then equality sign in (3.23) holds automatically.

(iii) If \(n = 2,\) then equality sign in (3.11) holds identically.

Proof. Let us consider that \(\mathcal{J}_M, M_1 \times_M M_2 \) be a \(C-\)totally real doubly warped product in a locally conformal almost cosymplectic manifold \(M(c)\) with pointwise constant \(\varphi-\)scetional curvature \(c.\) Then from Gauss equation, we find

\[2\rho = \left(\frac{c - 3 \delta^2}{4}\right)(n - 1) + n^2 ||H||^2 - ||h||^2. \tag{3.12}\]

Now, we consider that

\[\delta = 2\rho - \left(\frac{c - 3 \delta^2}{4}\right)(n + 1)(n - 2) - \frac{n^2(n - 2)}{n - 1} ||H||^2. \tag{3.13}\]

Then from (3.12) and (3.13), it follows that

\[n^2 ||H||^2 = (n - 1) \left[ ||h||^2 + \delta - \left(\frac{c - 3 \delta^2}{2}\right) \right]. \tag{3.14}\]

Let \(\{e_1, e_2, \ldots, e_n\} \) be an orthonormal frame, the above equation takes the following form

\[\sum_{r=n+1}^{2m+1} \sum_{j=1}^{n} h_{rj}^{n+1} = (n - 1) \left[ \delta + \sum_{r=n+1}^{2m+1} \sum_{j=1}^{n} (h_{rj}^{n+1})^2 + \sum_{r=n+1}^{2m+1} \sum_{l, j=1}^{n} (h_{rl}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{l, j=1}^{n} (h_{rl}^{n+1})^2 - \left(\frac{c - 3 \delta^2}{2}\right) \right], \]

which implies that

\[\left[h_{11}^{n+1} + \sum_{i=2}^{n_1} h_{ii}^{n+1} + \sum_{l=n+1}^{n} h_{ll}^{n+1}\right]^2 = \delta + \left(h_{11}^{n+1}\right)^2 + \sum_{i=2}^{n_1} \left(h_{ii}^{n+1}\right)^2 + \sum_{l=n+1}^{n} \left(h_{ll}^{n+1}\right)^2 - \sum_{2 \leq i \leq n_1} h_{ii}^{n+1} h_{ii}^{n+1}
- \sum_{n_1+1 \leq i \leq n} h_{ii}^{n+1} h_{ii}^{n+1} + \sum_{i=1}^{n} \sum_{j=1}^{n} (h_{ij}^{n+1})^2 + \sum_{r=n+1}^{2m+1} \sum_{l, j=1}^{n} (h_{rl}^{n+1})^2 - \left(\frac{c - 3 \delta^2}{2}\right). \tag{3.15}\]

Let us consider that \(a_1 = h_{11}^{n+1}, a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1} \) and \(a_3 = \sum_{l=n+1}^{n} h_{ll}^{n+1}.\) Then from Lemma 2.1 and equation (3.15), we get

\[\frac{\delta}{2} - \left(\frac{c - 3 \delta^2}{2}\right) + \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \left(\frac{1}{2}\right) \sum_{r=n+1}^{2m+1} \sum_{l, j=1}^{n} (h_{rl}^{n+1})^2 \leq \sum_{2 \leq i \leq n_1} h_{ii}^{n+1} h_{ii}^{n+1} + \sum_{n_1+1 \leq i \leq n} h_{ii}^{n+1} h_{ii}^{n+1}. \tag{3.16}\]

with equality holds in (3.16) if and only if

\[\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{l=n+1}^{n} h_{ll}^{n+1}. \tag{3.17}\]

On the other hand, from (3.16) and (2.13), we have

\[K(e_1 \wedge e_{n+1}) \geq \sum_{r=n+1}^{2m+1} \sum_{j \in P_{n+1}} (h_{1j}^{n+1})^2 + \left(\frac{1}{2}\right) \sum_{r=n+1}^{2m+1} \sum_{j \in P_{n+1}} (h_{1j}^{n+1})^2 + \sum_{r=n+1}^{2m+1} \sum_{j \in P_{n+1}} (h_{1j}^{n+1})^2. \]
respectively, we arrive at

\[ + \frac{1}{2} \sum_{r=r+1}^{2m+1} \sum_{i,j} (h'_{ij})^2 + \frac{1}{2} \sum_{r=r+1}^{2m+1} \sum_{i,j=1}^{n+1} (h'_{ij})^2 + \frac{\delta}{2}, \]

where \( P_{1n+1} = \{1, \cdots, n\} - \{1, n_1 + 1\} \). Thus, it implies that

\[ K(e_1 \wedge e_{n+1}) \geq \frac{\delta}{2}. \] (3.18)

Since, \( M = \varepsilon_1, M_1 \times_f M_2 \) is a \( C \)-totally real doubly warped product submanifold, we have \( \nabla_XZ = \nabla_ZX = (X \ln f_1)Z + (Z \ln f_2)X \), for any unit vector fields \( X \) and \( Z \) tangent to \( M_1 \) and \( M_2 \), respectively. Then from (2.15), (3.13) and (3.18), the scalar curvature derives as:

\[ \rho \leq \frac{1}{f_1} \{(\nabla_{e_1} e_1)f_1 - e_1^2f_1\} + \frac{1}{f_2} \{(\nabla_{e_2} e_2)f_2 - e_2^2f_2\} + \frac{n^2(n-2)}{2(n-1)} ||H||^2 + \left( \frac{c - 3\delta^2}{4} \right)(n+1)(n-2). \] (3.19)

Let the equality holds in (3.19), then all leaving terms in (3.16) and (3.18), we obtain the following conditions, i.e.,

\[ h'_{ij} = 0, \quad h'_{r_{mi+1}} = 0, \quad h'_{r_{ji}} = 0, \quad \text{where} \quad i \neq j, \quad \text{and} \quad r \in \{n+1, \cdots, 2m+1\}; \]
\[ h'_{1f} = h'_{1\text{tot}}, \quad h'_{ji} = 0, \quad \text{and} \quad h'_{11} + h'_{n_1n_1+1} = 0, \quad \text{where} \quad i, j \in P_{1n+1}, \quad r = n + 2, \cdots, 2m + 1. \] (3.20)

Similarly, we extend the relation (3.19) as follows

\[ \rho \leq \frac{1}{f_1} \{(\nabla_{e_1} e_1)f_1 - e_1^2f_1\} + \frac{1}{f_2} \{(\nabla_{e_2} e_2)f_2 - e_2^2f_2\} + \frac{n^2(n-2)}{2(n-1)} ||H||^2 + \left( \frac{c - 3\delta^2}{4} \right)(n+1)(n-2), \] (3.21)

for any \( \alpha = 1, \cdots, n_1 \) and \( \beta = n_1 + 1, \cdots, n \). Taking the summing up \( \alpha \) from 1 to \( n_1 \) and up \( \beta \) from \( n_1 + 1 \) to \( n_2 \) respectively, we arrive at

\[ n_1n_2\rho \leq \frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} + \frac{n^2n_1n_2(n-2)}{2(n-1)} ||H||^2 + \left( \frac{c - 3\delta^2}{4} \right)n_1n_2(n+1)(n-2). \] (3.22)

Similarly, the equality sign holds in (3.22) identically. Thus the equality sign in (3.19) holds for each \( \alpha \in \{1, \cdots, n_1\} \) and \( \beta \in \{n_1 + 1, \cdots, n\} \). Then we get the following;

\[ h'_{ni} = 0, \quad h'_{ij} = 0, \quad h'_{ri} = 0, \quad \text{where} \quad i \neq j, \quad \text{and} \quad r \in \{n + 1, \cdots, 2m + 1\}; \]
\[ h'_{r_{ni}} = h'_{r_{ij}} = 0, \quad \text{and} \quad h'_{r_{ni}} + h'_{r_{ij}} = 0, \quad \text{where} \quad i, j \in P_{1n+1}, \quad r = n + 2, \cdots, 2m + 1. \] (3.23)

Moreover, if \( n = 2 \). Then \( n_1 = n_2 = 1 \). thus from (2.15), we get \( \rho = \Delta_1 f_1 + \Delta_2 f_2 \). Hence the equality in (3.11) holds, which proves the theorem completely. \( \square \)

Now, we also have the following applications of Theorem 3.2.

**Remark 3.4.** If either \( f_1 = 1 \) or \( f_2 = 1 \) in Theorem 3.2, then we get following corollary for a \( C \)-totally real warped product.

**Corollary 3.5.** Let \( \tilde{M}(c) \) be a \((2m+1)-\)dimensional locally conformal almost cosymplectic manifold and \( \phi : M = M_1 \times_f M_2 \to \tilde{M}(c) \) be an isometric immersion of an \( n \)-dimensinal \( C \)-totally real warped product into \( \tilde{M}(c) \) such that \( c \) is pointwise constant \( \varphi \)-sectional curvature. Then

\[ \Delta f_{n_1 f} \geq \rho - \frac{n^2(n-2)}{2(n-1)} ||H||^2 - \left( \frac{c - 3\delta^2}{4} \right)(n+1)(n-2), \] (3.24)

where \( n_1 = \dim M_1, \quad i = 1, 2 \) and \( \Delta \) is the Laplacian operator on \( M_1 \).
that c is pointwise constant
\( \phi \)
the warping functions are harmonics. Then M is not minimal in \( \widetilde{M} \).

We, also have the following special cases of Theorem 3.2.

**Corollary 3.6.** Let \( \phi : M = f_1 \times f_2 \to \widetilde{M}(c) \) be an n-dimensional C–totally real doubly warped product into a locally conformal almost cosymplectic manifold \( \widetilde{M}(c) \) with c, a pointwise constant \( \phi \)–sectional curvature such that the warping functions are harmonics. Then M is not minimal in \( \widetilde{M} \) with inequality

\[
c > \frac{4\rho}{(n+1)(n-2)} + 3\delta^2.
\]

**Corollary 3.7.** Let \( \phi : M = f_1 \times f_2 \to \widetilde{M}(c) \) be an n-dimensionsal C–totally real doubly warped product into a locally conformal almost cosymplectic manifold \( \widetilde{M}(c) \) with c, a pointwise constant \( \phi \)–sectional curvature. Suppose that the warping functions \( f_1 \) and \( f_2 \) of M are the eigenfunctions of Laplacians on \( M_1 \) and \( M_2 \) with corresponding eigenvalues \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), respectively. Then M is not minimal in \( \widetilde{M} \) with inequality

\[
c \geq \frac{4\rho}{(n+1)(n-2)} + 3\delta^2.
\]

**Corollary 3.8.** Let \( \phi : M = f_1 \times f_2 \to \widetilde{M}(c) \) be an n-dimensional C–totally real doubly warped product into a locally conformal almost cosymplectic manifold \( \widetilde{M}(c) \) with c, a pointwise constant \( \phi \)–sectional curvature. Suppose that one of the warping function is harmonic and other one is eigenfunction of the Laplacian with corresponding eigenvalue \( \lambda > 0 \). Then M is not minimal in \( \widetilde{M} \) with inequality

\[
c \geq \frac{4\rho}{(n+1)(n-2)} + 3\delta^2.
\]

If we combine both Theorem 3.1 and Theorem 3.2, then we get the following result.

**Theorem 3.3.** Assume that \( \phi : M = f_1 \times f_2 \to \widetilde{M}(c) \) be an isometric immersion of an n-dimensionsal C–totally real doubly warped product into a \( (2m+1) \)-dimensional locally conformal almost cosymplectic manifold \( \widetilde{M}(c) \) such that c is pointwise constant \( \phi \)–sectional curvature. Then

\[
\rho - \frac{n^2(n-2)}{2(n-1)} \|H\|^2 - \left( \frac{c - 3\delta^2}{4} \right)(n+1)(n-2) \leq \left( \frac{\Delta f_1}{n_1} \right)^2 + \left( \frac{\Delta f_2}{n_2} \right)^2 \leq \frac{n^2}{4n_1n_2} \|H\|^2 + \left( \frac{c - 3\delta^2}{4} \right).
\]

**Remark 3.5.** Theorem 3.3 represent an upper and lower bounds for warping functions of a C–totally real doubly warped product.

**Remark 3.6.** If either \( f_1 = 1 \) or \( f_2 = 1 \) in Theorem 3.3, then we get following corollary for a C–totally real warped product.

**Corollary 3.9.** Assume that \( \phi : M = f_1 \times f_2 \to \widetilde{M}(c) \) be an isometric immersion of an n-dimensionsal C–totally real doubly warped product into a \( (2m+1) \)-dimensional locally conformal almost cosymplectic manifold \( \widetilde{M}(c) \) such that c is pointwise constant \( \phi \)–sectional curvature. Then

\[
\rho - \frac{n^2(n-2)}{2(n-1)} \|H\|^2 - \left( \frac{c - 3\delta^2}{4} \right)(n+1)(n-2) \leq \left( \frac{\Delta f}{n} \right)^2 \leq \frac{n^2}{4n_1n_2} \|H\|^2 + \left( \frac{c - 3\delta^2}{4} \right).
\]

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