Geometric aspects of CR-warped product submanifolds of T-manifolds

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In this paper, we study CR-warped product submanifolds of T-manifolds. We prove that the CR-warped product submanifolds with invariant fiber are trivial warped products and provide a characterization theorem of CR-warped products with anti-invariant fiber of T-manifolds. Moreover, we develop an inequality of CR-warped product submanifolds for the second fundamental form in terms of warping function and the equality cases are considered. Also, we find a necessary and sufficient condition for compact oriented CR-warped products turning into CR-products of T-space forms.

Keywords: CR-warped products; inequalities; compact CR-warped products; T-manifolds; T-space forms.

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1. Introduction

It is well known that the notion of warped product with negative curvature was initiated by Bishop and O’Neill [4]. Afterwards, Chen [8, 9] gave the idea of nontrivial CR-warped product of Kaehler manifold and established a geometric inequality for extrinsic invariant. However, many research papers appeared, exploring the existence and non-existence of warped product submanifolds in different kinds of ambient manifolds [1, 2, 10, 15, 16, 19, 21].

On the other hand, CR-submanifolds of a T-manifold were defined and studied by Calin [7]. In the present paper, our study generalize the results of Chen [8, 9] under the special cases in the setting of T-manifolds. We show that the warped

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product in the form $M = N_\perp \times f N_T$ is trivial where $N_\perp$ is an anti-invariant submanifold and $N_T$ is an invariant submanifold of a $T$-manifold $\bar{M}$. We give a characterization theorem when the characteristic vector fields are tangent to base manifold of the CR-warped product submanifold of type $M = N_T \times f N_\perp$. For these warped products, we derive a geometric inequality for the second fundamental form in terms of warping functions and give that $N_T$ is $\varphi$-minimal invariant submanifold into $\tilde{M}$ of $M = N_T \times f N_\perp$. Moreover, if we consider that $M$ is compact oriented submanifold without boundary, then we establish necessary and sufficient conditions for CR-warped products to become simply CR-products in $T$-space forms which are generalizing the same condition of complex space form and cosymplectic space forms.

2. Preliminaries

Let $\tilde{M}$ be a $(2n + s)$-dimensional differentiable manifold of class $C^\infty$ endowed with a $\varphi$-structure of rank $2n$. According to Blair [6], the $\varphi$-structure $\varphi$ is said to be a complemented frame if there exist structure vector fields $\xi^1, \xi^2, \ldots, \xi^s$ and their dual 1-forms $\eta_1, \eta_2, \ldots, \eta_s$ such that

$$\varphi^2 X = -X + \sum_{p=1}^{s} \eta_p(X) \otimes \xi^p, \quad (2.1)$$

$$\eta_p(\xi^q) = \delta_p^q, \quad \varphi(\xi^p) = 0, \quad \eta_p \circ \varphi = 0, \quad (2.2)$$

where $\delta_p^q$ denotes the Kronecker delta and $p, q = 1, 2, \ldots, s$. A manifold $\tilde{M}$ is said to have a metric $\varphi$-structure if there exists a Riemannian metric $g$ such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{p,q=1}^{s} \eta_p(X) \eta_q(Y) \quad (2.3)$$

for any $X, Y \in \Gamma(T\tilde{M})$. In this case,

$$g(\varphi X, Y) = -g(X, \varphi Y). \quad (2.4)$$

The fundamental 2-form $F$ on $\tilde{M}$ is given by

$$F(X, Y) = g(X, \varphi Y). \quad (2.5)$$

An almost contact metric manifold $\tilde{M}$ is said to be a $K$-manifold if the fundamental 2-form is closed and the metric $\varphi$-structure is normal [6], that is,

$$S_{\varphi}(X, Y) + 2 \sum_{p=1}^{s} d\eta_p(X, Y) \xi^p = 0, \quad (2.6)$$

where $S_{\varphi}(X, Y)$ denotes the Nijenhuis tensor with respect to the tensor field $\varphi$. A $K$-manifold with $d\eta_p = 0$ for each $p = 1, 2, \ldots, s$ is said to be a $T$-manifold.
Let \( \nabla \) be the Levi-Civitas connection with respect to the metric tensor \( g \) on a \( T \)-manifold \( \tilde{M} \), then we have [5]

\[
\begin{align*}
(a) \quad (\nabla_X \varphi)Y &= 0, \\
(b) \quad \nabla_X \xi^p &= 0
\end{align*}
\] (2.7)

for each \( X, Y \in \Gamma(TM) \), where \( TM \) denotes the tangent bundle of \( \tilde{M} \).

Assume \( M \) to be isometrically immersed into almost contact metric manifold \( \tilde{M} \) with induced Riemannian metric \( g \). Then Gauss and Weingarten formulas are given by

\[
\begin{align*}
\nabla_X Y &= \nabla_X Y + h(X, Y), \\
\nabla_X N &= -A_N X + \nabla^\perp_X N,
\end{align*}
\] (2.8)

where \( \nabla \) and \( \nabla^\perp \) are induced Riemannian connection on the tangent bundle \( TM \) and \( T \perp M \), respectively, such that for every \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \). Moreover, \( h \) and \( A_N \) are the second fundamental form and the shape operator (corresponding to the normal vector field \( N \)), respectively, for the immersion of \( M \) into \( \tilde{M} \). They are related as [16]

\[
g(h(X, Y), N) = g(A_N X, Y),
\] (2.10)

where \( g \) denotes the Riemannian metric on \( \tilde{M} \) as well as induced on \( M \). For any \( X \in TM \), we write

\[
\varphi X = PX + FX,
\] (2.11)

where \( PX \) is the tangential component and \( FX \) is the normal component of \( \varphi X \). Similarly, for any \( N \in \Gamma(T^\perp M) \), we write

\[
\varphi N = tN + fN,
\] (2.12)

where \( tN \) is the tangential component and \( fN \) is the normal component of \( \varphi N \). From Eqs. (2.7)(b) and (2.8), we obtain

\[
\begin{align*}
(a) \quad \nabla_X \xi^p &= 0, \\
(b) \quad h(X, \xi^p) &= 0
\end{align*}
\] (2.13)

for each \( X \in \Gamma(TM) \) and \( p = 1, 2, \ldots, s \). For submanifolds tangent to the structure vector fields \( \xi_1, \xi_2, \ldots, \xi_s \), we define semi-invariant submanifold as follows.

**Definition 2.1.** A submanifold \( M \) tangent to \( \xi_1, \xi_2, \ldots, \xi_s \) is called a contact CR-submanifold if it admits a pair of differentiable distributions \( D \) and \( D^\perp \) such that \( D \) is invariant and its orthogonal complementary distribution \( D^\perp \) is anti-invariant, i.e. \( TM = D \oplus D^\perp \oplus \langle \xi \rangle \) with \( f(D_p) \subseteq D_p \) and \( f(D^\perp) \subseteq T^\perp p M \) for every \( p \in M \).

In particular, the contact CR-submanifold \( M \) tangent to \( \xi_1, \xi_2, \ldots, \xi_s \), is invariant if \( D^\perp = \{0\} \) and anti-invariant if \( D = \{0\} \). Let \( M \) be an \( m \)-dimensional CR-submanifold of a \( T \)-manifold \( \tilde{M} \). Then, \( F(T_p M) \) is a subspace of \( T_p^\perp M \). Then for every \( p \in M \), there exists an invariant subspace \( \mu_p \) of \( T_p \tilde{M} \) such that

\[
T_p \tilde{M} = T_p M \oplus F(T_p M) \oplus \mu_p.
\] (2.14)
3. Geometry of Warped Product CR-Submanifolds

Let \((N_1, g_1)\) and \((N_2, g_2)\) be two Riemannian manifolds and \(f\), a positive differentiable function on \(N_1\). The warped product of \(N_1\) and \(N_2\) is the product manifold \(N_1 \times_f N_2 = (N_1 \times N_2, g)\) furnished with \(g = g_1 + f^2 g_2\). More explicitly, if \(U\) is tangent to \(M = N_1 \times_f N_2\) at \((p, q)\), then

\[
\|U\|^2 = \|d_\pi_1 U\|^2 + f^2(p)\|d_\pi_2 U\|^2, \tag{3.1}
\]

where \(\pi_i (i = 1, 2)\) are the canonical projections of \(N_1 \times N_2\) on \(N_1\) and \(N_2\), respectively. The following lemmas give some basic formulas on warped product submanifolds of \(T\)-manifold.

**Lemma 3.1** [4]. Let \(M = N_1 \times_f N_2\) be warped product manifold. If \(X, Y \in \Gamma(TN_1)\) and \(V, W \in \Gamma(TN_2)\), then

(i) \(\nabla_X Y \in \Gamma(TN_1)\),

(ii) \(\nabla_X V = \nabla_Y X = (\frac{X f}{f}) V\),

(iii) \(\nabla_V W = \nabla_W V = \frac{g(V, W)}{f} \nabla f\),

where \(\nabla'\) is the component of \(\nabla V W \in \Gamma(TN_1)\) and \(\nabla f\) is the gradient vector field of the warping function \(f\), defined by \(g(\nabla f, U) = U f\) for all \(U \in \Gamma(TM)\).

From (ii) of the above lemma, we can see that

\[
\nabla_X V = \nabla_Y X = (X \ln f) V \tag{3.2}
\]

for any \(X \in \Gamma(TN_1)\) and \(V \in \Gamma(TN_2)\). From the above lemma, we can see that if \(M = N_1 \times_f N_2\) be a warped product manifold, then \(N_1\) is totally geodesic and \(N_2\) is totally umbilical submanifold of \(M\), respectively. The warped product \(N_1 \times_f N_2\) is said to be trival if the warping function \(f\) is constant. In the following section, we shall investigate warped product CR-submanifolds of a \(T\)-manifold.

Throughout the section, \(M\) is a warped product submanifold of a \(T\)-manifold. If \(N_T\) and \(N_\perp\) are invariant and anti-invariant submanifolds of a \(T\)-manifold \(M\), then their warped product may be given by one of the following forms:

(i) \(M = N_\perp \times_f N_T\),

(ii) \(M = N_T \times_f N_\perp\).

The structure vector fields \(\xi^p\) either tangent to the invariant submanifold \(N_T\) or tangent to the anti-invariant submanifold \(N_\perp\) for all \(p = 1, 2, \ldots, s\). We start with the case when \(\xi^p\) tangent to \(N_\perp\) for all \(p = 1, 2, \ldots, s\).

**Theorem 3.1.** Every proper warped product CR-submanifold of type \(N_\perp \times_f N_T\) where \(N_\perp\) is an anti-invariant submanifold and \(N_T\) is an invariant submanifold of a \(T\)-manifold \(\tilde{M}\) such that \(\xi^\alpha\) tangent to \(N_\perp\) for all \(\alpha = 1, 2, \ldots, s\) is a simply Riemannian product manifold.
Proof. Assume that $M = N_\perp \times f N_T$ be a warped product CR-submanifold of a $T$-manifold $M$ with $\xi^\alpha \in TN_\perp$ for all $\alpha = 1, 2, \ldots, s$, then for any $X \in TN_T$ and $Z \in TN_\perp$, in particular, $Z = \xi$ in (3.2), then
\[
\nabla_X \xi^\alpha = (\xi^\alpha \ln f)X.
\]
From (2.13), we obtain that $\xi^\alpha \ln f = 0$ for all $\alpha = 1, 2, \ldots, s$. (3.3)

Now, taking the product with $X$ in (3.2), we get
\[
g(\nabla_X Z, X) = (Z \ln f) \|X\|^2. \quad (3.4)
\]
On the other hand, by (2.4) and (2.8), we have
\[
g(\nabla_X Z, X) = g(\nabla_X Z, X) = g(\varphi \nabla_X Z, \varphi X) + \sum_{\alpha, \beta = 1}^{s} \eta_\alpha(\nabla_X Z)\eta_\beta(X).
\]
As we considered $\xi$ tangent to $N_\perp$ for all $\alpha = 1, 2, \ldots, s$, we get
\[
g(\nabla_X Z, X) = g(\varphi \nabla_X Z, \varphi X).
\]
Then from tensorial equation (2.7) of $T$-manifold, we obtain
\[
g(\nabla_X Z, X) = g(\nabla_X Z, \varphi X).
\]
Thus, from (2.9) and (2.10), we get
\[
g(\nabla_X Z, X) = -g(A_F Z, \varphi X) = -g(h(X, \varphi X), FZ). \quad (3.5)
\]
The above equation takes the form with the account of Eq. (3.4)
\[
(Z \ln f) \|X\|^2 = -g(h(X, \varphi X), FZ). \quad (3.6)
\]
Replacing $X$ by $\varphi X$ in (3.6) and then using (2.1) and (2.4), and the fact that $\xi$ is tangent to $N_\perp$ for all $\alpha = 1, 2, \ldots, s$, we get
\[
(Z \ln f) \|X\|^2 = g(h(X, \varphi X), FZ). \quad (3.7)
\]
From Eqs. (3.6) and (3.7), we obtain that
\[
(Z \ln f) \|X\|^2 = 0. \quad (3.8)
\]
Thus, from Eqs. (3.3) and (3.8), it follows that $f$ is constant on $N_\perp$. This proves the theorem completely. \square

Now, the other case, i.e. $N_T \times f N_\perp$ with $\xi^\alpha \in TN_T$ for all $p = 1, 2, \ldots, s$ is dealt with the following results.

Lemma 3.2. Assume that $M = N_T \times f N_\perp$ is a nontrivial CR-warped product submanifold of $T$-manifold. Then
\[
g(\nabla_Z W, \varphi X) = -(\varphi X \ln f)g(Z, W), \quad (3.9)
g(h(X, Z), \varphi W) = -(\varphi X \ln f)g(Z, W) \quad (3.10)
\]
for any $X \in \Gamma(D \oplus \xi)$ and $Z, W \in \Gamma(D^\perp)$. 1750067-5
Proof. Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold with the structure vector fields $\xi^p$ tangent to $N_T$ for all $p = 1, 2, \ldots, s$, then for any $X \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_\perp)$, we have

$$g(\nabla ZW, \varphi X) = g(\nabla ZW, \varphi X) = -g(\nabla Z\phi X, W).$$

Then from (2.4), we obtain

$$g(\nabla ZW, \varphi X) = -g(\nabla Z\varphi W, X).$$

Using (2.9) and (2.10), we get

$$g(\nabla ZW, \varphi X) = g(A_{\varphi W} Z, X) = g(h(X, Z), FW).$$

Then from Eqs. (3.9) and (3.11), we obtain

$$g(h(X, Z), \varphi W) = -(\varphi X \ln f)g(Z, W).$$

This is the final result of lemma. This completes proof of the lemma.

Theorem 3.2. Every proper CR-submanifold $M$ of a $T$-manifold $\tilde{M}$ is locally a contact CR-warped product if and only if

$$A_{\varphi Z} X = -(\varphi X \mu) Z, \quad X \in \Gamma( D \oplus \langle \xi \rangle), \quad Z \in \Gamma( D^\perp)$$

for some function $\mu$ on $M$ satisfying $W(\lambda) = 0$ for each $W \in \Gamma(D^\perp)$.

Proof. Let us consider that $M = N_T \times_f N_\perp$ is a CR-warped product submanifold of a $T$-manifolds. Then first part directly follows from (3.10) of Lemma 3.2 with case $\mu = \ln f$.

Conversely, suppose $M$ is a CR-submanifold of $\tilde{M}$ and satisfying Eq. (3.12), then

$$g(h(X, Y), \varphi Z) = g(A_{\varphi Z} X, Y) = -(\varphi X \mu) g(Y, Z) = 0.$$ 

Using (2.5) and the fact that $\tilde{M}$ is a $T$-manifold, we obtain

$$g(\varphi \nabla X Y, Z) = -g(\nabla X \varphi Y, Z) = 0.$$ 

That is,

$$g(\nabla X Y, Z) = 0.$$ 

This means that $D \oplus \langle \xi \rangle$ is integrable and its leaves are totally geodesic in $M$. If $N_\perp$ be a leaf of $D^\perp$ and $h^\perp$ be the second fundamental form of the immersion of
CR-warped products of $T$-manifolds

$N_\perp$ into $M$, then for any $Z, W \in \Gamma(D_{\perp})$, we have

$$g(h^\perp(Z, W), \varphi X) = g(\nabla_W Z, \varphi X) = g(\tilde{\nabla}_W Z, \varphi X) = -g(\varphi \tilde{\nabla}_W Z, X).$$

Using the characteristic equation of $T$-manifold, we get

$$g(h^\perp(Z, W), \varphi X) = -g(\tilde{\nabla}_W \varphi Z, X) = g(A_{\varphi Z} W, X).$$

Then from (2.10), we drive

$$g(h^\perp(Z, W), \varphi X) = g(h(X, W), \varphi Z), \quad \text{or}$$

$$g(h^\perp(Z, W), \varphi X) = g(A_{\varphi Z} X, W).$$

Thus, from (3.12), we obtain

$$h^\perp(Z, W) = -g(Z, W)\nabla \lambda, \quad (3.13)$$

which implies that $N_\perp$ is totally umbilical in $M$ with the mean curvature vector $H = -\nabla \lambda$. Moreover, as $Z \lambda = 0$ for all $Z \in \Gamma(D_{\perp})$, that is, the mean curvature is parallel on $N_\perp$, this shows that $N_\perp$ is extrinsic sphere. Applying the result of Hiepko [14], we obtain that $M$ is locally a CR-warped product submanifold $N_T \times f N_{\perp}$ with the warping function $f = e^\lambda$ of a $T$-manifold $\tilde{M}$. Hence, the theorem is proved.

Proposition 3.1. Let $M = N_T \times f N_{\perp}$ be a CR-warped product submanifold of $T$-manifolds such that the structure vector field $\xi^p$ is tangent to $N_T$ for all $p = 1, 2, \ldots, s$. Then

$$g(h(\varphi X, Z), \varphi W) = (X \ln f)g(Z \cdot W), \quad (3.14)$$

$$g(h(\varphi X, Y), \varphi Z) = g(h(X, Y), \varphi Z) = 0, \quad (3.15)$$

$$g(h(X, X), \tau) = -g(h(\varphi X, \varphi X), \tau) \quad (3.16)$$

for any $X, Y \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_{\perp})$. Moreover, $\tau \in \Gamma(\mu)$.

**Proof.** From (2.8), (2.11), (2.7)(a) and Lemma 3.1(ii), we derive

$$g(h(\varphi X, Z), \varphi W) = g(\tilde{\nabla}_Z \varphi X, \varphi W) = g(\tilde{\nabla}_Z X, \varphi W) = (X \ln f)g(Z \cdot W),$$

which is the result (3.14). On the other parts, from (2.3), (2.4) and (2.7(a)), we obtain

$$g(h(\varphi X, Y), \varphi Z) = g(\tilde{\nabla}_Y \varphi X, \varphi Z) = g(\tilde{\nabla}_Y X, Z) - \sum_{p=1}^{s} \eta^p(\tilde{\nabla}_Y X)\eta^p(Z).$$

Thus, from the facts that $\xi$ is tangent to $N_T$ and $N_T$ is totally geodesic in $M$ and then using (2.8), we get the required result (3.15). In a similar way, we obtain (3.16). This completes the proof of the proposition. □
Proposition 3.2. Assume that $M = N_T \times_f N_\perp$ be a CR-warped product submanifold of $T$-manifolds such that $N_T$ is invariant submanifolds tangent to $\xi^1, \xi^2, \ldots, \xi^s$. Then

$$||h(X, Z)||^2 = g(h(\varphi X, Z), \varphi h(X, Z)) + (\varphi X \ln f)^2||Z||^2$$

for any $X, Y \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_\perp)$.

Proof. By the definition of the norm and (2.3), we derive

$$||h(X, Z)||^2 = g(\varphi h(X, Z), \varphi h(X, Z)) + \sum_{p=1}^{s} \eta^{2p}(h(X, Z)).$$

Using (2.8), (2.7)(a) and Lemma 3.1(ii), we obtain

$$||h(X, Z)||^2 = g(\tilde{\nabla}_Z \varphi X, \varphi h(X, Z)) - (\varphi X \ln f)g(\varphi h(X, Z), Z).$$

Again from (2.8) and Lemma 3.1(ii), we arrive at

$$||h(X, Z)||^2 = g(h(\varphi X, Z), \varphi h(X, Z)) - (\varphi X \ln f)g(h(X, Z), \varphi Z).$$

Thus, using (3.10) in the last term of right-hand side of the above equation, we get the required result. This completes the proof of the proposition. $\blacksquare$

Theorem 3.3. Let $\tilde{M}$ be a $2m + s$-dimensional $T$-manifold and $M = N_T \times_f N_\perp$ be a $n$-dimensional CR-warped product submanifold of $\tilde{M}$ such that $N_T$ is $2\alpha + s$-dimensional invariant submanifold tangent to $\xi^p$. Then

(i) The squared norm of the second fundamental form is given by

$$||h||^2 \geq 2\beta||\nabla \ln f||^2, \quad (3.17)$$

where $\beta$ is dimension of anti-invariant submanifold $N_\perp$ and $p = 1, 2, \ldots, s$.

(ii) The equality holds in (3.17), then $N_T$ is totally geodesic and $N_\perp$ is totally umbilical submanifolds of $\tilde{M}$, respectively. Moreover, $N_\perp$ is minimal submanifold of $\tilde{M}$.

Proof. Suppose that $M = N_T \times_f N_\perp$ be a $n = 2\delta + \beta + s$-dimensional CR-warped product submanifold into $2m + s$-dimensional $T$-manifold $\tilde{M}$, where $N_T$ is a $2\alpha + s$-dimensional invariant submanifold tangent to $\xi^p$ for all $p = 1, 2, \ldots, s$ and $N_\perp$ is anti-invariant submanifold of dimension $\beta$. Then we consider that $\{e_1 = \tilde{e}_1, \ldots, e_\alpha = \tilde{e}_\alpha, e_{\alpha+1} = \varphi \tilde{e}_1, \ldots, e_{2\alpha} = \varphi \tilde{e}_\alpha, e_{2\alpha+1} = \xi^1, \ldots, e_{2\alpha+s} = \xi^p\}$ and $\{e_{2\alpha+s+1} = \tilde{e}_1, \ldots, e_{2\alpha+s+\beta} = \tilde{e}_\beta\}$ are orthonormal frames for integral manifolds $N_T$ and $N_\perp$ of $D$ and $D^\perp$, respectively. Moreover, the orthonormal frames for the normal sub-bundle $\varphi D^\perp$ and $\mu$ are $\{e_{n+1} = \tilde{e}_1 = \varphi \tilde{e}_1, \ldots, e_{n+\beta} = \tilde{e}_\beta = \varphi \tilde{e}_\beta\}$.
and \( \{e_{n+\beta+1}, \ldots, e_{2m+}\} \), respectively. Thus, in the definition of the second fundamental form, we have

\[
||h||^2 = ||h(D, D)||^2 + ||h(D^+, D^+)||^2 + 2||h(D, D^+)||^2.
\]

The above equation can be expressed as

\[
||h||^2 = \sum_{r=n+1}^{2m+} \sum_{i,j=1}^{2n+} g(h(e_i, e_j), e_r)^2 + \sum_{r=n+1}^{2m+} \sum_{i,j=1}^{2n+} g(h(e_i, e_j), e_r)^2
\]

\[
+ 2\sum_{r=n+1}^{2m+} \sum_{i,j=1}^{2n+} g(h(e_i, e_j), e_r)^2.
\]

Leaving all terms except third term and using adapted frame in (3.18), then (3.18) takes new form

\[
||h||^2 \geq 2\sum_{r=n+1}^{n+\beta} \sum_{i,j=1}^{2n+} g(h(e_i, e_j), e_r)^2 + 2\sum_{r=n+1}^{n+\beta} \sum_{i,j=1}^{2n+} g(h(e_i, e_j), e_r)^2
\]

\[
+ 2\sum_{r=n+1}^{n+\beta+1} \sum_{i,j=1}^{2n+} g(h(e_i, e_j), e_r)^2.
\]

The second term of the right-hand side of the above equation is identically zero by using the fact that \( h(\xi^p, Z) = 0 \) for all \( p = 1, 2, \ldots, s \). Now, consider only first term and using the adapted frame, we derive

\[
||h||^2 \geq 2\sum_{r,j=1}^{r,j=1} g(h(\bar{e}_i, \bar{e}_j), \varphi \bar{e}_r)^2 + 2\sum_{r,j=1}^{r,j=1} g(h(\varphi \bar{e}_i, \bar{e}_j), \varphi \bar{e}_r)^2.
\]

Thus, from (3.10) and (3.14), we obtain

\[
||h||^2 \geq 2\sum_{r,j=1}^{\alpha} (\varphi \bar{e}_i \ln f)^2 g(\bar{e}_r, \bar{e}_j)^2 + 2\sum_{r,j=1}^{\alpha} (\bar{e}_i \ln f)^2 g(\bar{e}_r, \bar{e}_j)^2.
\]

Adding and subtracting the same terms \( \xi^p \ln f \) in (3.20) and from (3.3), for all \( p = 1, 2, \ldots, s \), we get

\[
||h||^2 \geq 2\beta \sum_{i=1}^{2n+} (\xi_i \ln f)^2.
\]

Hence, the inequality (3.17) holds. If the equality holds in (3.17), then by leaving the terms in (3.18), we get

\[
h(D, D) = 0, \quad h(D^+, D^+) = 0
\]

for \( D = D^+ \). Also from (3.19), we obtain

\[
h(D, D^+) \subseteq \varphi D^+.
\]
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Then from (3.21) and (3.22), it is easy to conclude that $N_T$ is totally geodesic in $\tilde{M}$ as it is totally geodesic in $M$ and $N_\perp$ is totally umbilical submanifold of $\tilde{M}$. Furthermore, $N_\perp$ is totally umbilical, then we can write $h(Z, W) = g(Z, W)H$ for $Z, W \in \Gamma(D \perp)$. Thus, from (3.21), we get $g(Z, W)H = 0$, which implies that $H = 0$, its means that $N_\perp$ is minimal submanifold. This completes the proof of the theorem.

Theorem 3.4 [20]. On a CR-warped product submanifold $M = N_T \times_f N_\perp$ of a $T$-manifold such that $N_T$ is invariant submanifold of dimension $n_1$ tangent to $\xi^1, \xi^2, \ldots, \xi^s$. Then $N_T$ is $\varphi$-minimal submanifold of $\tilde{M}$.

Proof. We skip the proof of above theorem due to similarity of proof in [20].


Let $M$ be an isometrically immersed submanifold into a $T$-manifold $\tilde{M}$. Then Gauss and Codazzi equations are defined, respectively, as

\[
(\tilde{R}(X, Y)Z)^T = R(X, Y)Z + Ah(X, Z)Y - Ah(Y, Z)X, \quad \text{and}
\]
\[
(\tilde{R}(X, Y)Z)^\perp = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z)
\]  

(4.1)

for any $X, Y, Z \in \Gamma(\tilde{M})$, where $(\tilde{R}(X, Y)Z)^T$ and $(\tilde{R}(X, Y)Z)^\perp$ are tangential and normal components of $(\tilde{R}(X, Y)Z)$, respectively. Moreover, the covariant derivative of the second fundamental form is given by

\[
(\nabla_X h)(Y, Z) = \nabla_Y Z h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
\]  

(4.2)

Now let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal frame for the tangent space $TM$, then for a differential function $f$, the Hessian form and Laplacian are defined as

\[
H^f(X, Y) = X(Y(f)) - (\nabla_X Y)f = g(\nabla_X \text{grad}f, Y) \quad \text{and}
\]
\[
\Delta f = \sum_{i=1}^{n} \{(\nabla_{e_i} e_i)f - e_i^2 f\} = -\sum_{i=1}^{n} H^f(e_i, e_i).
\]  

(4.3)

Furthermore, on a compact orientate Riemannian manifold without boundary, from the concept of integration theory on manifolds, we have [2]

\[
\int_M \Delta f dV = 0,
\]  

(4.4)

where $dV$ is defined as a volume element of $M$. If $\tilde{M}$ has constant $\varphi$-sectional curvature $c$. Then $\tilde{M}$ is called a $T$-space form and denoted by $\tilde{M}(c)$. Thus, the
Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}(c)$ is defined as

$$
(\tilde{R}(X, Y)Z, W) = \frac{c}{4} \left( g(Y, W)g(X, Z) - g(X, W)g(Y, Z) \\
- g(X, Z) \sum_{p, q=1}^{s} \eta^p(Y)\eta^q(W) - g(Y, W) \sum_{p, q=1}^{s} \eta^p(X)\eta^q(Z) \\
+ g(X, W) \sum_{p, q=1}^{s} \eta^p(Y)\eta^q(Z) + g(Y, Z) \sum_{p, q=1}^{s} \eta^p(X)\eta^q(W) \\
+ \left( \sum_{p, q=1}^{s} \eta^p(Y)\eta^q(W) \right) \left( \sum_{p, q=1}^{s} \eta^p(X)\eta^q(Z) \right) \\
- \left( \sum_{p, q=1}^{s} \eta^p(Y)\eta^q(Z) \right) \left( \sum_{p, q=1}^{s} \eta^p(X)\eta^q(W) \right) \\
+ g(\varphi Y, X)g(\varphi Z, W) - g(\varphi X, Z)g(\varphi Y, W) \\
+ 2g(X, \varphi Y)g(\varphi Z, W) \right)
$$

(4.5)

for any $X, Y, Z, W \in \Gamma(\tilde{M})$ [3]. If we set $s = 0$, then $T$-space form generalized to complex space form [19] and if $s = 1$, the above formula generalized the Riemannian curvature tensor in cosymplectic space forms [2].

**Proposition 4.1.** On a CR-warped product submanifold $M = N_T \times f N_{\perp}$ of $T$-manifold $\tilde{M}$. Then

$$
g(\tilde{R}(X, \varphi X)Z, \varphi Z) + 2||h(X, Z)||^2 = (H^{\ln f}(X, X) + H^{\ln f}(\varphi X, \varphi X) \\
+ 2(\varphi X \ln f)^2)||Z||^2
$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_{\perp})$.

**Proof.** Making use of Codazzi equation and Riemannian curvature tensor $\tilde{R}$, we obtain

$$
g(\tilde{R}(X, \varphi X)Z, \varphi Z) = g((\tilde{\nabla}_X h)(\varphi X, Z), \varphi Z) - g((\tilde{\nabla}_{\varphi X} h)(\varphi X, Z), \varphi Z)
$$

for $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_{\perp})$. Thus, from the covariant derivative of the second fundamental form (4.2), (2.9) and Lemma 3.1(ii), we derive

$$
g(\tilde{R}(X, \varphi X)Z, \varphi Z) = g(\tilde{\nabla}_X h(\varphi X, Z), \varphi Z) - g(h(\nabla_X \varphi X, Z), \varphi Z) \\
- (X \ln f)g(h(\varphi X, Z), \varphi Z) - g(\tilde{\nabla}_{\varphi X} h(X, Z), \varphi Z) \\
+ g(h(\nabla_{\varphi X} X, Z), \varphi Z) + (\varphi X \ln f)g(h(X, Z), \varphi Z).
$$
The first and fourth terms follow by the property of the derivative of vector field on the right-hand side of the above equation and hence, we arrive at

\[ g(\tilde{R}(X, \phi X)Z, \phi Z) = Xg(h(\phi X, Z), \phi Z) - g(h(\phi X, Z), \tilde{\nabla}_X \phi Z) - g(h(\nabla_X \phi X, Z), \phi Z) - (X \ln f)g(h(\phi X, Z), \phi Z) + g(h(\nabla_{\phi X} X, Z), \phi Z) + (\phi X \ln f)g(h(X, Z), \phi Z) - \phi Xg(h(X, Z), \phi Z) + g(h(X, Z), \tilde{\nabla}_\phi X \phi Z). \]

Thus, from (3.14) Lemma 3.1(ii) and (2.7)(a), we obtain

\[ g(\tilde{R}(X, \phi X)Z, \phi Z) = X(\ln f)||Z||^2 + 2(\ln f)^2||Z||^2 - g(h(\phi X, Z), \phi \nabla_X Z) - g(h(\nabla_X \phi X, Z), \phi Z) - (\ln f)^2||Z||^2 + 2g(h(\nabla_{\phi X} X, Z), \phi Z) - (\phi X \ln f)^2||Z||^2 + \phi X(\phi X \ln f)||Z||^2 + 2(\phi X \ln f)^2||Z||^2 + g(h(X, Z), \phi \nabla_{\phi X} Z). \]

Since \( \nabla_{\phi X} X, \nabla_X \phi X \in \Gamma(TN_T) \) and the fact that \( N_T \) is totally geodesic in \( M \), from (2.8) and (3.10), we get

\[ g(\tilde{R}(X, \phi X)Z, \phi Z) = X(\ln f)||Z||^2 + (\ln f)^2||Z||^2 + \phi X(\phi X \ln f)||Z||^2 - g(h(\phi X, Z), \phi \nabla_X Z) - g(h(\nabla_X \phi X, Z), \phi Z) - (\ln f)^2||Z||^2 + 2g(h(\nabla_{\phi X} X, Z), \phi Z) - (\phi X \ln f)^2||Z||^2 + \phi X(\phi X \ln f)||Z||^2 + 2(\phi X \ln f)^2||Z||^2 + g(h(X, Z), \phi \nabla_{\phi X} Z). \]

Using (3.10), Lemma 3.1(ii) and Proposition 3.2, we arrive at

\[ g(\tilde{R}(X, \phi X)Z, \phi Z) = X(\ln f)||Z||^2 + (\ln f)^2||Z||^2 + \phi X(\phi X \ln f)||Z||^2 - (\ln f)^2||Z||^2 - 2||h(X, Z)||^2 + 2(\phi X \ln f)^2||Z||^2 + (\phi X \ln f)^2||Z||^2 + (\phi X \ln f)^2||Z||^2 - (\phi X \ln f)^2||Z||^2, \]

which implies that

\[ g(\tilde{R}(X, \phi X)Z, \phi Z) = X(\ln f)||Z||^2 + \phi X(\phi X \ln f)||Z||^2 - 2||h(X, Z)||^2 + 2(\phi X \ln f)^2||Z||^2 - \nabla_X \phi X \ln f||Z||^2. \]

Thus, from the definition of Hessian form, we get the required result. This completes the proof of the proposition. \( \Box \)
Proposition 4.2. Let $\widetilde{M}(c)$ be a $T$-space form and $M = N_T \times_f N_\perp$ be a CR-warped product submanifold of $\widetilde{M}(c)$. Then
\[
||h(X, Z)||^2 = \frac{1}{2} \left( H^{\ln f}(X, X) + H^{\ln f}(\varphi X, \varphi X) \right. \\
+ (\varphi X \ln f)^2 + \frac{c}{4} ||X||^2 \left. ||Z||^2 \right)
\]
for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\perp)$.

Proof. Taking account that the ambient manifold is a $T$-space form, from (4.4) and the fact that $\xi^p$ is tangent to $N_T$ for all $p = 1, 2, \ldots, s$, we obtain
\[
g(\tilde{R}(X, \varphi X)Z, \varphi Z) = \frac{c}{4} ||X||^2 ||Z||^2.
\]
Hence, using Proposition 4.1 in the above relation, we get the required result. \[\square\]

Theorem 4.1. Let $M = N_T \times_f N_\perp$ be a compact CR-warped product submanifold in a $T$-space form $\widetilde{M}(c)$ such that $N_T$ is invariant submanifold tangent to $\xi^2, \xi^3, \ldots, \xi^s$ of dimension $n_1 = 2a + s$ and $N_\perp$ is anti-invariant of dimension $n_2 = \beta$-dimensional submanifold. Then $M$ is a CR-product if and only if
\[
\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h_{\mu}(e_i, e_j)||^2 = \frac{\alpha \cdot \beta \cdot c}{4},
\]
where $h_{\mu}$ denote the components of $h$ in $\Gamma(\mu)$.

Proof. From (4.3) and using adapted frame of CR-warped product submanifolds, we obtain
\[
\Delta(\ln f) = -2 \sum_{i=1}^{\alpha} H^{\ln f}(e_i, e_i) - \sum_{j=1}^{\beta} H^{\ln f}(e_j, e_j) - \sum_{p=1}^{s} g(\nabla_{\xi^p} \text{grad} \ln f, \xi^p).
\]
Since ambient space $\widetilde{M}$ is $T$-manifold and the fact that $\text{grad} \ln f \in \Gamma(TN_T)$, which means that $g(\nabla_{\xi^p} \text{grad} \ln f, \xi^p) = 0$ for all $p = 1, 2, \ldots, s$, we have
\[
\Delta(\ln f) = -2 \sum_{i=1}^{\alpha} H^{\ln f}(e_i, e_i) - \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} H^{\ln f}(\varphi e_i, \varphi e_j) - \sum_{j=1}^{\beta} g(\nabla_{e_j} \nabla \ln f, e_j).
\]
Now taking into account that $M$ is compact orientate submanifold, from (4.4), we get
\[
\int_M \left( \sum_{i=1}^{\alpha} \{ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \} + \sum_{j=1}^{\beta} e_j g(\nabla \ln f, e_j) \right) dV
\]
\[= \int_M \left( \sum_{j=1}^{\beta} g(\nabla \ln f, \nabla e_j) \right) dV,
\]
where $dV$ is a volume element over integration of compact submanifolds $M$. Thus, from gradient function of $\ln f$, we derive

$$
\int_M \left( \sum_{i=1}^{\alpha} (H^1 f(e_i, e_i) + H^1 f(\varphi e_i, \varphi e_i)) + \sum_{j=1}^{\beta} (e_j (e_j \ln f) - \nabla e_j e_j \ln f) \right) dV = 0.
$$

Using Lemma 3.1(ii) and the fact that $\nabla \ln f \in \Gamma(TN_T)$, we obtain

$$
\int_M \left( \sum_{i=1}^{\alpha} (H^1 f(e_i, e_i) + H^1 f(\varphi e_i, \varphi e_i)) + \beta ||\nabla \ln f||^2 \right) dV = 0. \quad (4.6)
$$

On the other hand, let $X = e_i$ and $Z = e_j$ for $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$, the second fundamental form can be expressed as

$$
h(e_i, e_j) = \sum_{r=1}^{\alpha} g(h(e_i, e_j), \varphi e_j) \varphi e_j + \sum_{r=\beta+1}^{2m+s} g(h(e_i, e_j), \tau e_r) \tau e_r,
$$

where $\tau \in \Gamma(\mu)$. Taking summation over $\alpha$ and $\beta$, we obtain

$$
\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h(e_i, e_j)||^2 = \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(e_i, e_j), \varphi e_j)^2 + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(e_i, e_j), \tau e_r)^2.
$$

Using (3.10) in the first term of right-hand side of the above equation, we get

$$
\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h(e_i, e_j)||^2 = \beta \sum_{i=1}^{\alpha} (\varphi e_i \ln f)^2 + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h_{\mu}(e_i, e_j)||^2. \quad (4.7)
$$

Summing up Proposition 4.2, we derive

$$
\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h(e_i, e_j)||^2 = \frac{\beta}{2} \sum_{i=1}^{\alpha} (H^1 f(e_i, e_i) + H^1 f(\varphi e_i, \varphi e_i))
$$

$$
+ \beta \sum_{i=1}^{\alpha} (\varphi e_i \ln f)^2 + \frac{c \cdot \alpha \cdot \beta}{4}. \quad (4.8)
$$

Thus from (4.3) and (4.8), we obtain

$$
2 \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h_{\mu}(e_i, e_j)||^2 = \sum_{i=1}^{\alpha} (H^1 f(e_i, e_i) + H^1 f(\varphi e_i, \varphi e_i)) + \frac{c \cdot \alpha \cdot \beta}{2}. \quad (4.9)
$$

Finally, using the above equation in (4.2), we get

$$
\int_M \left( \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h_{\mu}(e_i, e_j)||^2 - \frac{c \cdot \alpha \cdot \beta}{4} + \beta ||\nabla \ln f||^2 \right) dV = 0. \quad (4.10)
$$

Suppose that $M$ be a compact CR-warped product submanifold with condition (4.1) holds. Then from (4.10), we get $||\nabla \ln f||^2 = 0$, which implies that $\text{grad} \ln f = 0$. 
meaning that the warping function $f$ is constant on $M$. Hence by hypothesis of warped product manifolds, $M$ is simply CR-product of invariant and anti-invariant submanifolds $N_T$ and $N_\perp$, respectively. Conversely, let $M$ be a compact CR-product in $T$-space form $\overline{M}(c)$. Then, the warping function $f$ must be constant on $M$, i.e. $\nabla \ln f = 0$. Thus, equality (4.10) implies the equality (4.1). This completes the proof of the theorem.

**Corollary 4.1.** Let $M = N_T \times_f N_\perp$ be a compact CR-warped product submanifold in a $T$-space form $\overline{M}(c)$ such that $N_T$ is invariant submanifold tangent to $\xi^2, \xi^3, \ldots, \xi^n$ of dimension $n_1 = 2\alpha + s$ and $N_\perp$ is anti-invariant $n_2$-dimensional submanifold. Then $M$ is a CR-product if and only if

$$H\ln f(e_i, e_i) + H\ln f(\varphi e_i, \varphi e_i) = 0,$$

where $H$ is a Hessian form and $1 \leq i \leq \alpha$.

**Proof.** The proof of the above corollary follows from (4.2).

**Remark 4.1.** Throughout the study in this paper, we see that if we set $s = 1$ in $T$-manifold, then all the results which we obtain in the present paper generalize the results for contact CR-warped product submanifolds in cosymplectic manifolds (see [2, 16, 21]).

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**References**

A. Ali & W. A. M. Othman