A Note On The Accuracy Of The Adomian Decomposition Method Applied To The Chaotic Lorenz System

Noor Fadiya Mohd Noor\textsuperscript{1}, Ishak Hashim\textsuperscript{2}, Mohd Salmi Md Noorani\textsuperscript{2}

\textsuperscript{1}Science and Mathematics Department
Faculty of Science and Education
Universiti Industri Selangor
Kampus Bestari Jaya
45600 Selangor, Malaysia
fadiya@unisel.zzn.com

\textsuperscript{2}School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi, Selangor, Malaysia
ishak.h@ukm.my

Abstract. In this paper, the Adomian decomposition method (ADM) is applied to the famous Lorenz system. The ADM yields an analytical solution in terms of a rapidly convergent infinite power series with easily computable terms. The ADM is treated as an algorithm for approximating the solutions of the Lorenz system in a sequence of time intervals (i.e. time steps). Comparisons between the decomposition solutions and seventh-eighth order Runge-Kutta (RK78) numerical solutions are made for various error tolerances. In particular, we look at the accuracy of the ADM for solving the chaotic Lorenz system.

2000 Mathematics Subject Classification: 30C45, Secondary 30C80

Key words and phrases: Adomian decomposition method, Seventh-eighth order Runge-Kutta method, Lorenz system, Chaos

1. Introduction

Comparative study on nonlinear systems of ordinary differential equations is often inconclusive because of the unavailability of a closed form analytic solution. Because of a lack of global solutions via any numerical method, interest in the Adomian decomposition method \cite{1, 2, 3} becomes widespread as shown in Olek \cite{12}, Shawagfeh and Adomian \cite{14}, Biazar \textit{et al.} \cite{4}, Biazar and Montazeri \cite{5} and in many other previous works \cite{6, 7, 9, 10, 12, 13, 14, 15, 16, 17, 18} for the ADM yields, without linearization, perturbation, transformation or discretization, an analytical solution in terms of a rapidly convergent infinite power series with easily computable terms (see \cite{8}).

Recently, the accuracy of the ADM method was tested with the numerical method of Runge-Kutta-Verner by Olek \cite{12} and Vadasz and Olek \cite{16} while Guellal \textit{et al.} \cite{7}, Shawagfeh and Kaya \cite{15} and Hashim \textit{et al.} \cite{8} compared the ADM with the Runge-Kutta method of order four (RK4). These studies emphasized the effect of time steps on the accuracy of the solutions. In
this paper, we shift the focus to include the effect of error tolerances in determining the accuracy as well as providing possible comparison between the ADM decomposition method and other numerical methods.

The accuracy of the Adomian decomposition method is investigated by comparing the method with the Runge-Kutta method of seventh-eighth order (RK78) at various error tolerances in solving the chaotic Lorenz system which is known as highly sensitive to initial conditions. Following the work by Hashim et al. [8], the step size is set to $\Delta t = 0.001$. The objective of this paper is therefore to give comparison between the ADM and RK78 solutions of the chaotic Lorenz system and to study the effect of time step imposed with various error tolerances on the accuracy of the ADM method.

2. Adomian Decomposition Solution to The Lorenz system

The famous Lorenz system of equations outlined in [11] is given as

\begin{align}
\frac{dx}{dt} &= \sigma(y-x), \\
\frac{dy}{dt} &= Rx - y - xz, \\
\frac{dz}{dt} &= xy + bz,
\end{align}

where $x$, $y$ and $z$ are respectively proportional to the convective velocity, the temperature difference between descending and ascending flows, and the mean convective heat flow, where $\sigma$, $b$ and the bifurcation parameter $R$ are real constants. Throughout this paper, we set $\sigma = 10$, $b = -8/3$ and consider the case $R = 28$ where the system exhibits chaotic behavior, and we vary the error tolerance from $10^{-6}$ up to $10^{-10}$ and $10^{-12}$ at time step $\Delta t = 0.001$.

Hence the explicit solution to the Lorenz system (2.1)–(2.3) as derived in [8] is

\begin{align}
x &= \sum_{m=0}^{\infty} a_m \frac{(t-t^*)^m}{m!}, \\
y &= \sum_{m=0}^{\infty} b_m \frac{(t-t^*)^m}{m!}, \\
z &= \sum_{m=0}^{\infty} c_m \frac{(t-t^*)^m}{m!},
\end{align}

where the coefficients are given by the recurrence relations,

\begin{align}
a_0 &= x(t^*), \quad b_0 = y(t^*), \quad c_0 = z(t^*), \\
a_m &= -\sigma a_{m-1} + \sigma b_{m-1}, \quad m \geq 1, \\
b_m &= Ra_{m-1} - b_{m-1} - (m-1)! \sum_{k=0}^{m-1} \frac{a_k c_{m-k-1}}{k!(m-k-1)!}, \quad m \geq 1, \\
k_m &= bc_{m-1} + (m-1)! \sum_{k=0}^{m-1} \frac{a_k b_{m-k-1}}{k!(m-k-1)!}, \quad m \geq 1.
\end{align}
The ADM method is treated as an algorithm to approximate the dynamical response in a sequence of time intervals \( (i.e. \) time step) \( [0, t_1), [t_1, t_2), \ldots, [t_{m-1}, T) \) such that the initial condition in \( [t^*, t_{m+1}) \) is taken to be the condition at \( t^* \) as first hinted in [1] and applied in [7, 12, 15, 16, 8]. For practical computations, a finite number of terms in the series (2.4)–(2.6) are used in a time step procedure given.

3. Results and discussion

The Adomian algorithm is coded in the computer algebra package Maple and we employ Maple’s built-in seventh-eighth-order Runge-Kutta procedure \( \text{rk78} \). The Maple environment variable \( \text{Digits} \) controlling the number of significant digits is set to 25 in all the calculations done in this paper. For a direct comparison with Guellal et al. [7] and Hashim et al. [8], we set \( \sigma = 10, b = -8/3 \) and take the initial conditions \( x(0) = -15.8, y(0) = -17.48 \) and \( z(0) = 35.64 \). The time range studied in this work is \( [0, 20] \) as considered in [7] and [8]. In particular, we focus on chaotic Lorenz system when \( R = 28 \) which is considered in [7, 8].

Since the system (2.1)–(2.3) with \( R = 28 \) and the other parameters given previously exhibits chaotic solutions and was accepted as highly sensitive to time step by Hashim et al. [8], the resulting solutions here are also expected to be sensitive to error tolerance. The differences between the RK78 solutions at error tolerances \( 10^{-6}, 10^{-10} \) and \( 10^{-12} \) are given in Table 1. Although the solutions of the chaotic system become less accurate as time increases, the implementation of the error tolerance \( 10^{-12} \) over the tolerance \( 10^{-10} \) does improve the accuracy of the RK78 solutions up to multiple of \( 10^{-4} \). In Table 2, the details of the differences between the 10-term decomposition solutions on \( \Delta t = 0.001 \) by Hashim et al. [8] against the RK78 solutions at the tolerances \( 10^{-6}, 10^{-10} \) and \( 10^{-12} \) are presented. From this table, we discover that the maximum difference between the decomposition solutions and the RK78 solutions at the tolerance \( 10^{-12} \) is of magnitude of order \( 10^{-5} \). These solutions reflect that at the error tolerance \( 10^{-12} \), the Runge-Kutta method of seventh-eighth order approximate the Adomian solutions up to 8-th time interval almost accurately than at the other two tolerances. This agrees with the right hand side figures in Figure 1 which obviously showed that the numerical method at the tolerance \( 10^{-12} \) approximate the Adomian solutions most accurately.

4. Conclusions

In this work, the Runge-Kutta method of seventh-eighth order was applied in generating solutions to the chaotic Lorenz system of equations. Comparison between the decomposition solutions and the RK78 solutions were made. The numerical solutions achieved are of comparable accuracy at time step \( \Delta t = 0.001, \) error tolerance \( 10^{-12} \). We note that the decomposition solutions applied to the chaotic Lorenz system by Hashim et al. [8] are highly accurate though they were computed much more simply in terms of algorithm, amount of computation and a lack of numerical techniques used.
Table 1. An accuracy determination of RK78 solutions for $R = 28$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\Delta x$</th>
<th>$\Delta y$</th>
<th>$\Delta z$</th>
<th>$\Delta x$</th>
<th>$\Delta y$</th>
<th>$\Delta z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.55E-07</td>
<td>1.92E-07</td>
<td>3.6E-07</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>4</td>
<td>2.115E-06</td>
<td>6.670E-07</td>
<td>3.970E-06</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>6</td>
<td>1.625E-05</td>
<td>2.638E-05</td>
<td>4.940E-06</td>
<td>2.000E-09</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>8</td>
<td>3.742E-05</td>
<td>5.978E-05</td>
<td>2.125E-05</td>
<td>4.000E-09</td>
<td>7.000E-09</td>
<td>0.000</td>
</tr>
<tr>
<td>10</td>
<td>3.462E-04</td>
<td>2.937E-03</td>
<td>2.217E-03</td>
<td>3.000E-08</td>
<td>3.300E-07</td>
<td>2.500E-07</td>
</tr>
<tr>
<td>12</td>
<td>1.995E-02</td>
<td>2.240E-02</td>
<td>2.741E-02</td>
<td>2.262E-06</td>
<td>2.551E-06</td>
<td>3.080E-06</td>
</tr>
<tr>
<td>14</td>
<td>1.995E-02</td>
<td>2.240E-02</td>
<td>2.741E-02</td>
<td>2.262E-06</td>
<td>2.551E-06</td>
<td>3.080E-06</td>
</tr>
<tr>
<td>16</td>
<td>1.813E-01</td>
<td>4.339E-02</td>
<td>2.835E-01</td>
<td>1.862E-05</td>
<td>7.000E-09</td>
<td>3.026E-05</td>
</tr>
<tr>
<td>18</td>
<td>2.394</td>
<td>4.142</td>
<td>1.440</td>
<td>2.486E-04</td>
<td>4.118E-04</td>
<td>2.253E-04</td>
</tr>
</tbody>
</table>

Figure 1. $R = 28$ on $\Delta t = 0.001$: $\text{tol} = 10^{-6}$ (left) vs $\text{tol} = 10^{-10}$ (centre) vs $\text{tol} = 10^{-12}$ (right)

5. Copyright

School of Mathematical Sciences USM, reserves the right to publish papers at this conference.

References

Table 2. Differences between 10-term decomposition and RK78 solutions for $R = 28$, $\Delta t = 0.001$ at various error tolerances.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\Delta x$</th>
<th>$\Delta y$</th>
<th>$\Delta z$</th>
<th>$\Delta x$</th>
<th>$\Delta y$</th>
<th>$\Delta z$</th>
<th>$\Delta x$</th>
<th>$\Delta y$</th>
<th>$\Delta z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.550E-07</td>
<td>1.920E-07</td>
<td>3.600E-07</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>4</td>
<td>2.115E-06</td>
<td>6.670E-07</td>
<td>3.970E-06</td>
<td>2.000E-09</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>6</td>
<td>1.625E-05</td>
<td>2.638E-05</td>
<td>4.940E-06</td>
<td>5.000E-09</td>
<td>8.000E-09</td>
<td>0.000</td>
<td>1.000E-09</td>
<td>1.000E-09</td>
<td>0.000</td>
</tr>
<tr>
<td>8</td>
<td>3.741E-05</td>
<td>5.977E-05</td>
<td>2.125E-05</td>
<td>4.000E-08</td>
<td>3.900E-07</td>
<td>2.900E-07</td>
<td>1.000E-08</td>
<td>6.000E-08</td>
<td>4.000E-08</td>
</tr>
<tr>
<td>10</td>
<td>3.461E-04</td>
<td>2.936E-03</td>
<td>2.217E-03</td>
<td>2.000E-07</td>
<td>1.400E-07</td>
<td>0.000</td>
<td>3.000E-08</td>
<td>2.000E-08</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>1.573E-04</td>
<td>1.573E-03</td>
<td>1.083E-03</td>
<td>1.540E-06</td>
<td>6.100E-07</td>
<td>4.600E-06</td>
<td>2.300E-07</td>
<td>9.000E-08</td>
<td>6.600E-07</td>
</tr>
<tr>
<td>14</td>
<td>1.163E-02</td>
<td>4.545E-03</td>
<td>3.472E-02</td>
<td>1.540E-06</td>
<td>6.100E-07</td>
<td>4.600E-06</td>
<td>2.300E-07</td>
<td>9.000E-08</td>
<td>6.600E-07</td>
</tr>
<tr>
<td>16</td>
<td>1.995E-02</td>
<td>2.239E-02</td>
<td>2.741E-02</td>
<td>2.644E-06</td>
<td>2.982E-06</td>
<td>3.600E-06</td>
<td>3.820E-07</td>
<td>4.310E-07</td>
<td>5.200E-07</td>
</tr>
<tr>
<td>18</td>
<td>1.813E-01</td>
<td>4.339E-02</td>
<td>2.835E-01</td>
<td>2.177E-05</td>
<td>3.305E-06</td>
<td>3.537E-05</td>
<td>3.148E-06</td>
<td>4.780E-07</td>
<td>5.110E-06</td>
</tr>
</tbody>
</table>


