DIFFERENTIAL SUBORDINATION PROPERTIES
OF CERTAIN ANALYTIC FUNCTIONS

RASHIDAH OMAR*1,†, SUZEINI ABDUL HALIM*§
and RABHA W. IBRAHIM*¶
*Institute of Mathematical Sciences, Faculty of Science
University of Malaya, Malaysia
†Faculty of Computer and Mathematical Sciences
MARA University of Technology, Malaysia
‡Ashidah@hotmail.com
§suzeini@um.edu.my
¶rabhaibrahim@yahoo.com

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This paper looks at subordination properties involving $1 + \beta z p' (z)$ and the class of Sokół–Stankiewicz starlike functions where condition on $\beta$ is determined so that the relation $1 + \beta z p'(z) \prec \sqrt{1 + z}$ would imply $p(z) \prec \sqrt{1 + z}$. Similar results are obtained for expressions of the form $1 + \beta z p'(z) p(z)$ and $1 + \beta z p'(z) p^2(z)$. These results are then applied to obtain other relations involving similar expressions.

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1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the open unit disk $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ and normalized by $f(0) = 0$, $f'(0) = 1$.

Let $\mathcal{C}$ and $\mathcal{S}^*$ respectively denote the class of convex and starlike functions. An analytic function $f$ is subordinate to an analytic function $g$, written $f(z) \prec g(z)$ ($z \in \mathbb{D}$), if there exists an analytic function $w$ in $\mathbb{D}$ such that $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. In particular, if $g$ is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ be an analytic function in a domain $E \subset \mathbb{C}^2$. Analytic function $p$ with $(p(z), zp'(z)) \in E(z \in \mathbb{D})$ is said to satisfy the first-order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z), \quad z \in \mathbb{D}, \quad \psi(p(0), 0) = h(0),$$

where $h$ is an analytic and univalent function in $\mathbb{D}$.
A univalent function \( q \) is called a dominant of the differential subordination of (1) if \( p \prec q \) for all \( p \) satisfying (1). A dominant \( \tilde{q} \) of (1) that satisfies \( \tilde{q} \prec q \) for all dominants \( q \) of (1) is called the best dominant of (1).

For \( c \in (0,1] \), Aouf et al. [3] defined the class \( S^*(q_c) \) as:

\[
S^*(q_c) = \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} \right|^2 - 1 < c, z \in D \right\}.
\]

It can be established that

\[
f \in S^*(q_c) \iff \frac{zf'(z)}{f(z)} < \sqrt{1 + cz} \quad (z \in D).
\]

Denote \( \Theta_c \) as the set of all points on the right half-plane such that the product of the distances from each point to the focuses \(-1\) and \(1\) is less than \( c\):

\[
\Theta_c := \{ w \in C : \text{Re}w > 0, |w^2 - 1| < c \}
\]

thus the boundary \( \partial\Theta_c \) is the right loop of the Cassinian ovals \((x^2 + y^2)^2 - 2(x^2 - y^2) = c^2 - 1\) and particularly, for \( c = 1\), the right half of the lemniscate of Bernoulli is obtained where \( S^*(q_1) \equiv SL^*\). The class of \( SL^* \) was introduced by Sokół and Stankiewicz in 1996 [21] and a function in this class is called a Sokół–Stankiewicz starlike function. Some properties of functions in class \( SL^* \) have been studied by [1, 17–20].

Quite a number of authors (see [8–10, 14 and 15]) have intensively studied properties of functions involving the expression \([1 + \frac{zf''(z)}{f'(z)}]{\frac{f''(z)}{f'(z)}}\) which can also be expressed as \([1 + \frac{zf''(z)}{f'(z)}]{\frac{f'(z)}{f''(z)}}\]. In fact, the order of starlikeness for functions in the class

\[
G_b := \left\{ f \in A : \left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{z}{f'(z)}} \right|^2 - 1 \prec b, 0 < b \leq 1, z \in D \right\}
\]

has been obtained by Silverman [16]. Obradović and Tuneski [11] improved the result by showing \( G_b \subset S^*[0, -b] \subset S^*[\frac{b}{1 + \sqrt{1 + b^2}}] \). Tuneski [22] obtained conditions for the inclusion \( G_b \subset S^*[A, B] \) to hold, where \( S^*[A, B] \) denote the class of Janowski starlike functions first introduced by Janowski [5] consisting of functions \( f \in A \) satisfying

\[
\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1).
\]

Rosihan et al. [2] unified the properties obtained in [22] by considering the general analytic function \( p(z) \) as \( \frac{zf'(z)}{f(z)} \). The condition is given as follows:

\[
1 + \frac{zp'(z)}{p^2(z)} \prec 1 + bz \Rightarrow p(z) \prec \frac{1 + Az}{1 + Bz}.
\]
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Let

\[ \frac{(zf(z))''}{f''(z)} - \frac{2zf'(z)}{f(z)} + \frac{(1-\alpha)z}{2-\alpha} \Rightarrow \left| \frac{z^2f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha \quad (0 \leq \alpha < 1). \]

Also, the expression can further be generalized in the following manner: \(1 + \beta zp(z)\) and \(1 + \beta zq(z)\) are subordinated to \(1 + \frac{\beta zp(z)}{p(z)}\) and \(1 + \frac{\beta zq(z)}{q(z)}\) implies \(p(z) < \frac{1 + A_p}{1 + B_p}\) for \(p(z) = \frac{z^2f'(z)}{f(z)^2}, f \in A\). Another special case of the above implications can be found in [13].

Nunokawa et al. [7] studied the criterion for a normalized analytic function to be univalent using the result \(1 + zp'(z) < 1 + z\) implies \(p(z) < 1 + z\) where \(p(z)\) is analytic in \(D\) and \(p(0) = 1\). In [2], conditions \(A, B, D\) and \(E\) were determined so that when \(1 + \beta zp'(z), 1 + \frac{\beta zp'(z)}{p(z)}\) and \(1 + \frac{\beta zp'(z)}{p(z)}\) are subordinated to \(1 + \frac{D_p}{1 + E_p}\), the relation \(p(z) < \frac{1 + A_p}{1 + B_p}\) holds true. Some applications using these properties have been obtained for analytic functions in the class of Janowski starlike functions. Furthermore in [1], the class of Sokół–Stankiewicz starlike functions was considered in obtaining conditions on \(\beta\) so that the above implications holds. In [12], the authors determined values of \(\beta\) so that the subordination of \(1 + \beta zp'(z), 1 + \frac{\beta zp'(z)}{p(z)}\) and \(1 + \frac{\beta zp'(z)}{p(z)}\) to \(1 + \frac{D_p}{1 + E_p}\) implies \(p(z) < \sqrt{1 + z}\). Motivated by these studies, this paper determines conditions for \(\beta\) by considering the classes of \(S^*(q_c)\) and \(SL^*\). The applications using these properties are also looked at.

2. Main Results

We first state the following lemma which is required to establish our results.

**Lemma 1** ([6]). Let \(q\) be univalent in \(D\) and let \(\varphi\) be analytic in a domain containing \(q(D)\). Let \(zq'(z)\varphi[q(z)]\) be starlike. If \(p\) is analytic in \(D\), \(p(0) = q(0)\) and satisfies \(zp'(z)\varphi[p(z)] < zq'(z)\varphi[q(z)]\) then \(p(z) < q(z)\) and \(q\) is the best dominant.

**Theorem 1.** Let \(p\) be an analytic function on \(D\), \(p(0) = 1\) and \(\beta_0 = 2\sqrt{2} \times (\sqrt{e + 1} - 1)\) where \(c \in (0, 1]\). If the function \(p\) satisfies the subordination

\[ 1 + \beta zp'(z) < \sqrt{1 + cz} \quad (\beta \geq \beta_0) \]

then

\[ p(z) < \sqrt{1 + z}. \]

**Proof.** Define the function \(q : D \rightarrow C\) by \(q(z) = \sqrt{1 + z}\) with \(q(0) = 1\). Since \(q(D) = \{w : |w^2 - 1| < 1\}\) is the interior of the right half of the lemniscate of Bernoulli, \(q(D)\) is a convex set and hence \(q\) is a convex function. The Alexander’s Theorem showed that \(zq'(z)\) is starlike function with respect to 0. It follows from Lemma 1 that,

\[ 1 + \beta zp'(z) < 1 + \beta zq'(z) \Rightarrow p(z) < q(z). \]
Corollary 1. Let $s(z) = \sqrt{1 + cz} \prec 1 + \beta zq'(z) = 1 + \frac{\beta z}{2\sqrt{1 + z}} = h(z)$.

Suppose $s(z) = \sqrt{1 + cz} = w$, since $s^{-1}(w) = \frac{w^2 - 1}{c}$ then we have

$$s^{-1}[h(z)] = \left[1 + \frac{\beta z}{2\sqrt{1 + z}}\right]^2 - 1$$

$$= \frac{1 + 2\left(\frac{\beta z}{2\sqrt{1 + z}}\right) + \left(\frac{\beta z}{2\sqrt{1 + z}}\right)^2 - 1}{c}$$

$$= \frac{1}{c} \left[\frac{\beta z}{2\sqrt{1 + z}}\right] \left[2 + \frac{\beta z}{2\sqrt{1 + z}}\right].$$

For $z = e^{i\theta}(\theta \in (-\pi, \pi))$, we obtain

$$|s^{-1}[h(z)]| = |s^{-1}[h(e^{i\theta})]| = \frac{1}{c} \left[\frac{\beta e^{i\theta}}{2\sqrt{1 + e^{i\theta}}}\right]^2 + \frac{\beta e^{i\theta}}{2\sqrt{1 + e^{i\theta}}}.$$  

Since the above expression is minimum when $2\sqrt{1 + e^{i\theta}} = 2\sqrt{2} \cos \frac{\theta}{2}$ is maximum and this occurs at $\theta = 0$. Thus

$$|s^{-1}[h(z)]| \geq \frac{1}{c} \left[\frac{\beta}{2\sqrt{2}}\right] \left[2 + \frac{\beta}{2\sqrt{2}}\right] = \frac{1}{c} \left[\left(1 + \frac{\beta}{2\sqrt{2}}\right)^2 - 1\right] \geq 1$$

for $\beta \geq 2\sqrt{2}(\sqrt{c + 1} - 1)$. Hence $D \subset s^{-1}[h(D)]$ or $s(D) \subset h(D)$ which implies $s(z) \prec h(z)$ and this proves the result.

By taking $p(z) = \frac{zf''(z)}{f'(z)}$ and $p(z) = f'(z)$, we have the following result using Theorem 1.

Corollary 1. Let $\beta_0 = 2\sqrt{2}(\sqrt{c + 1} - 1)$ where $c \in (0, 1]$ and $f \in A$.

(i) If $f$ satisfies the following

$$1 + \beta \frac{zf''(z)}{f'(z)} \left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f'(z)} + 1\right) \prec \sqrt{1 + cz}, \quad (\beta \geq \beta_0)$$

then $f \in SL^*$.  

(ii) If $1 + \beta zf''(z) \prec \sqrt{1 + cz}$ then $f'(z) \prec \sqrt{1 + z}$.

Theorem 2. Let $\beta_0 = 4(\sqrt{c + 1} - 1)$ and $c \in (0, 1]$. If

$$1 + \beta \frac{zp''(z)}{p(z)} \prec \sqrt{1 + cz} \quad \text{then} \quad p(z) \prec \sqrt{1 + z}, \quad (\beta \geq \beta_0).$$
Proof. Let \( q(z) = \sqrt{1 + z} \), \( q(0) = 1 \). Elementary calculation will show that \( \frac{\beta z q'(z)}{q(z)} = \frac{\beta z}{2(1+z)} \) is starlike. Application of Lemma 1 will deduce

\[
1 + \beta \frac{z p'(z)}{p(z)} < 1 + \beta \frac{z q'(z)}{q(z)} \Rightarrow p(z) < q(z),
\]

provided we show that

\[
s(z) = \sqrt{1 + cz} < 1 + \beta \frac{z q'(z)}{q(z)} = 1 + \frac{\beta z}{2(1+z)} = h(z).
\]

Similar as previous method, we obtain

\[
s^{-1}[h(z)] = \left[ 1 + \frac{\beta z}{2(1+z)} \right] - 1 = \frac{1}{c} \left[ \frac{2\beta z}{2(1+z)} + \left( \frac{\beta z}{2(1+z)} \right)^2 \right].
\]

For \( z = e^{i\theta}, \theta \in (-\pi, \pi) \),

\[
|s^{-1}[h(e^{i\theta})]| = \left| \frac{1}{c} \left[ \frac{2\beta e^{i\theta}}{2(1+e^{i\theta})} + \left( \frac{\beta e^{i\theta}}{2(1+e^{i\theta})} \right)^2 \right] \right| = \frac{1}{c} \left[ \frac{\beta e^{i\theta}}{2(1+e^{i\theta})} \right] \left[ 2 + \frac{\beta e^{i\theta}}{2(1+e^{i\theta})} \right].
\]

Since \( \max |1 + e^{i\theta}| = 2 \cos \frac{\theta}{2} \), this implies the minimum of the above expression is attained at \( \theta = 0 \).

Then \( |s^{-1}[h(z)]| \geq \frac{\beta}{2c} [2 + \frac{\beta}{4}] = \frac{1}{4} [(1 + \frac{\beta}{4})^2 - 1] \geq 1 \) for \( \beta \geq 4(\sqrt{c+1} - 1) \). Hence \( s(z) \prec h(z) \). \( \square \)

Applying Theorem 2 and letting \( p(z) = \frac{zf''(z)}{f'(z)} \) in (i) and \( p(z) = \frac{zf'(z)}{f(z)} \) in (ii), the following results are obtained.

Corollary 2. Let \( \beta_0 = 4(\sqrt{c+1} - 1) \) and \( f \in A \).

(i)

\[
1 + \beta \left[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right] \prec \sqrt{1 + cz} \Rightarrow f \in SL^* \quad (\beta \geq \beta_0).
\]

(ii)

\[
1 + \beta \left[ \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right] \prec \sqrt{1 + cz} \Rightarrow \frac{zf'(z)}{f(z)} \prec \sqrt{1 + \beta} \quad (\beta \geq \beta_0).
\]

Theorem 3. Let \( \beta_0 = 4\sqrt{2(\sqrt{c+1} - 1)} \). If

\[
1 + \beta \frac{z p'(z)}{p^2(z)} \prec \sqrt{1 + cz} \Rightarrow p(z) \prec \sqrt{1 + \beta} \quad (\beta \geq \beta_0).
\]
Proof. For \( q(z) = \sqrt{1 + z} \), \( \frac{2q'(z)}{q'(z)} \) is starlike. Lemma 1 can then imply the following relation

\[
1 + \beta \frac{2p'(z)}{p^2(z)} < 1 + \beta \frac{2q'(z)}{q^2(z)} \Rightarrow p(z) < q(z).
\]

Writing \( h(z) = 1 + \beta \frac{2q'(z)}{q^2(z)} = 1 + \frac{\beta z}{2(1 + z)^2} \) we then have

\[
s^{-1}[h(z)] = \left[ \frac{1 + \frac{\beta z}{2(1 + z)^2}}{c} \right]^2 - 1
\]

\[
= \frac{1}{c} \left[ \frac{\beta z}{2(1 + z)^2} \left( 2 + \frac{\beta z}{2(1 + z)^2} \right) \right].
\]

In a similar manner to previous cases, for \( z = e^{i\theta}, \theta \in (-\pi, \pi) \):

\[
|s^{-1}[h(z)]| \geq \frac{1}{c} \left| (1 + \frac{\beta z}{2(1 + z)^2})^2 - 1 \right| \geq 1 \text{ for } \beta \geq 4\sqrt{2}(\sqrt{c + 1} - 1). \]

Hence \( D \subset s^{-1}[h(D)] \) implies \( s(z) < h(z) \).

By setting \( p(z) = \frac{zf'(z)}{f(z)} \) in Theorem 3, we have the following corollary.

Corollary 3. Let \( \beta_0 = 4\sqrt{2}(\sqrt{c + 1} - 1) \) and \( f \in A \),

\[
1 - \beta + \beta \left[ 1 + \frac{zf'(z)}{f(z)} \right] < \sqrt{1 + cz} \Rightarrow f \in SL^* \quad (\beta \geq \beta_0).
\]

Remark 1. For the special case of \( c = 1 \), all the above theorems and corollaries are reduced to the results obtained in [1].

References


