Existence of fractional differential chains and factorizations based on transformations

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In this work, we deal with the existence of the fractional integrable equations involving two generalized symmetries compatible with nonlinear systems. The method used is based on the Bäcklund transformation or B-transformation. Furthermore, we shall factorize the fractional heat operator in order to yield the fractional Riccati equation. This is done by utilizing matrix transform Miura type and matrix operators, that is, matrices whose entries are differential operators of fractional order. The fractional differential operator is taken in the sense of Riemann–Liouville calculus. Copyright © 2014 John Wiley & Sons, Ltd.

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1. Introduction

The class of fractional differential equations has rendered superb device to characterize many complex phenomena in almost all areas of science, engineering, food processing, computer sciences, and social sciences. It includes a variety of equations such as fractional diffusion equations, fractional diffie-integral equations, mixed fractional equations, multilinear fractional differential equations, nonlinear fractional differential equations, fuzzy fractional differential equation, and boundary value problems. This class has gone through substantial growth in the past two decades and was used in solving numerous numerical and analytical problems [1–10]. More recently, the complex fractional differential equations were investigated, by the first author along with others [11–20], in the sense of the Srivastava–Owa differential operator, its generalization and modification in a complex domain, and its behavior under the Hadamard fractional differential operator as well as its application in the geometric function theory and theory of univalent functions [18–20]. Other applications of fractional order systems using various fractional differential equations were considered widely in recent years and appeared in many fields such as chaotic phenomena, control systems, dynamical systems synchronization, and other related concepts [21–24]. Many fractional dynamical systems and integer systems are viewed as complex bifurcation and chaotic phenomena. The discrete fractional calculus and discrete fractional equation are two other related topics that were investigated and utilized in physical problems, image proceeding, signal processing, and other area of computer science in the past few years by the first author and a few others [25–28].

The main aim of this paper is to study the mixture of these two classes of functions, namely, continuous fractional differential equations and discrete fractional equations. Our argument is based on the chain concept. A chain is an integrable fractional differential–difference equation joining one continuous variable and one discrete variable. We deal with the existence of the fractional integrable equation involving two generalized symmetries compatible nonlinear systems using B-transformation. We then, by utilizing a matrix transform Miura type and matrix operators, that is, matrices whose entries are differential operators of fractional order, shall factorize the fractional heat operator in order to obtain the fractional Riccati equation. The fractional operators are taken in the sense of the Riemann–Liouville operators.

2. Main results

In this section, we investigate a generalized transformation as well as the B-transformation when a fractional system of equations is computed through the elimination of shifts of two integrable chains.

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2.1. \( B \)-transformation

Suppose the fractional chain

\[
\partial_x^\alpha v_{m,n} = \frac{1}{v_m+1,n - v_{m-1,n}},
\]

(1)

where \( v_{m,n} = v(m,n; \zeta, \chi) \) is a function depending on discrete and continuous variables \((m,n) \in \mathbb{Z}^2\) (discrete variables) and \((\zeta, \chi) \in \mathbb{R}^2\) (continuous variables).

The fractional derivative \( \partial_x^{\alpha} \) stands as the Riemann–Liouville differential operator

\[
\partial_x^{\alpha} \phi(\zeta) = (d/d\chi) \int_0^\chi \frac{(\chi - t)^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) dt.
\]

We let the lowest order of (1) to be given by

\[
\partial_x^{\alpha} v_{m,n} = \frac{v_{m+2,n} - v_{m-2,n}}{(v_m+1,n - v_{m-1,n})^2 (v_{m+2,n} - v_{m,n}) (v_{m,n} - v_{m-2,n})},
\]

(2)

where \( \partial_x^{\alpha} \partial_x^{\alpha} v_{m,n} = \partial_x^{2 \alpha} v_{m,n} \).

Now, we utilize (1) and its shifted types to impose a system of fractional differential equations.

From (1), it follows that

\[
v_{m-2,n} = v_{m,n} - \frac{1}{\partial_x^{\alpha} v_{m-1,n}}, \quad v_{m-1,n} = v_{m+1,n} - \frac{1}{\partial_x^{\alpha} v_{m,n}}, \quad v_{m+2,n} = v_{m,n} + \frac{1}{\partial_x^{\alpha} v_{m+1,n}}.
\]

(3)

Substitute (3) into (2) to receive the nonlinear system

\[
\partial_x^{\alpha} v = \partial_x^{\alpha} \left( \partial_x^{\alpha} v \right) + 2 \left( \partial_x^{\alpha} v \right)^2 \partial_x^{2 \alpha} w
\]

\[
\partial_x^{\alpha} w = -\partial_x^{\alpha} \left( \partial_x^{\alpha} w \right) + 2 \left( \partial_x^{\alpha} w \right)^2 \partial_x^{2 \alpha} v,
\]

(4)

where \( v := v_{m,n} \) and \( w := v_{m+1,n} \). Once again from (1), we have

\[
(V_{m,n}, v_{m+1,n}) \rightarrow (V_{m+1,n}, v_{m+2,n}), \quad (V_{m-1,n}, v_{m,n}) \rightarrow (V_{m,n}, v_{m+1,n}).
\]

Thus, we conclude the respective transformations

\[
\begin{pmatrix} v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} w \\ v + \frac{1}{\partial_x^{\alpha} w} \end{pmatrix}
\]

(5)

and

\[
\begin{pmatrix} v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} w - \frac{1}{\partial_x^{\alpha} v} \\ v \end{pmatrix}.
\]

Moreover, Equations (1) and (2) can be realized as the discrete equation of the form

\[
(V_{m,n} - V_{m+1,n+1}) (V_{m+1,n} - V_{m,n+1}) + \sigma = \rho,
\]

(6)

where \( \sigma \) and \( \rho \) are constants. This equation is well known as the KdV-type equation and pKdV-type equation. In addition, equations (1) and (2) can be recognized as symmetries of (7). Hence, this implies that there is a function \( G \) of \((\zeta, \chi)\) satisfying the relations

\[
\partial_x^{\alpha} G = 0, \quad \partial_x^{2 \alpha} G = 0,
\]

(7)

where

\[
G := (v_{m,n} - v_{m+1,n+1})(v_{m+1,n} - v_{m,n+1}) + \sigma - \rho.
\]

Because Equations (1) and (2) do not include shifts with respect to \( n \), we may have the quantities

\[
\phi = v_{m,n+1}, \quad \psi = v_{m+1,n+1}.
\]

In addition, \( \phi \) and \( \psi \) satisfy system (4), that is,

\[
\partial_x^{\alpha} \phi = \partial_x^{\alpha} \left( \partial_x^{\alpha} \phi \right) + 2 \left( \partial_x^{\alpha} \phi \right)^2 \partial_x^{2 \alpha} \psi
\]

\[
\partial_x^{\alpha} \psi = -\partial_x^{\alpha} \left( \partial_x^{\alpha} \psi \right) + 2 \left( \partial_x^{\alpha} \psi \right)^2 \partial_x^{2 \alpha} \phi.
\]

(8)

We conclude that equation (7) can be read as

\[
(v - \psi)(w - \phi) = K, \quad K := \rho - \sigma.
\]

(9)

Later, we may assume that \( K \) is a B-transformation parameter. Furthermore, Equation (7) can be written in the up-down shifted form with respect to \( m \).
\[(v_{m+1,n} - v_{m+2,n+1}) (v_{m+2,n} - v_{m+1,n+1}) = K, \quad (10)\]

and
\[(v_{m-1,n} - v_{m,n+1}) (v_{m,n} - v_{m-1,n+1}) = K, \quad (11)\]

It follows from Equation (1) that
\[v_{m+2,n} = v + \frac{1}{\partial_x v}, \quad v_{m+2,n+1} = \phi + \frac{1}{\partial_x \psi}, \]
and
\[v_{m-1,n} = w + \frac{1}{\partial_x v}, \quad v_{m-1,n+1} = \psi + \frac{1}{\partial_x \psi}. \]

Substituting the last two assertions in (10) and (11), we obtain the relations
\[\left( w - \phi - \frac{1}{\partial_x v} \right) \left( v - \psi - \frac{1}{\partial_x w} \right) = K, \quad (12)\]
and
\[\left( w - \phi - \frac{1}{\partial_x v} \right) \left( v - \psi - \frac{1}{\partial_x \psi} \right) = K. \quad (13)\]

Hence, we have shown that the system of fractional differential Equations (13) and (14) formalizes the B-transformation (4) where we may refer that Equation (10) together with Equation (13) reduce to (14). Subsequently, any relations of (10), (13), and (14) imply B-transformation. Also, we note that the chains ought to perform the generalized symmetries of nonlinear system of fractional differential equation. Furthermore, these chains consider the existence of recursion relations involving the fractional operators. This contraction of fractional differential chains leads to the exact solutions of the nonlinear system.

2.2. Generalized transformation

Next, we deal with the periodicity of fractional system depending on fractional chains by assuming the reduction generalized transformation
\[v_m \rightarrow v_{\mu m + \nu n}, \quad \mu, \nu \in \mathbb{Z}. \]

Note that \(\mu\) and \(\nu\) impose the periodicity constraint \(v_{m,n} = v_{m-\nu,n+\mu}\). We shall discuss the case \(\mu = 1\). By employing the aforementioned reduction into Equation (7), we receive the relation
\[(v_m - v_{m+\nu+1}) (v_{m+1} - v_{m+\nu+1}) = K, \quad (14)\]

where \(\nu\) is an arbitrary integer number. It is clear that the chains (1) and (2) pull out this reduction for all \(\nu \in \mathbb{Z}\); thus, they preserve the symmetries of (15) during the transformation
\[v_{m+i,0} \rightarrow v_{m+i}. \]

By letting
\[v := v_k \quad \text{and} \quad w := v_{m+1}, \]
Equation (15) satisfies the transformation
\[(v - \psi)(w - \phi) = K, \]
where
\[v_{m+\nu} = \phi, \quad v_{m+\nu+1} = \psi. \]

Thus, we may persist to drive (13) and (14).

Example 2.1

Let \(\mu = 1\) and \(\nu = 2\). Equation (7) reduces into the difference equation
\[(v_m - v_{m+3}) (v_{m+1} - v_{m+2}) = K, \quad (15)\]
and chain (1) befits
\[\partial_x^2 v_m = \frac{1}{v_{m+1} - v_{m-1}}. \quad (16)\]
Elementary computations from (16–17) with \( i = 0, 1, 2 \) yield the nonlinear system of fractional differential equations

\[
\begin{align*}
\partial^\alpha_x v_0 &= \frac{(v_1 - v_0)(v_2 - v_0)}{h}, \\
\partial^\alpha_x v_1 &= \frac{1}{v_2 - v_0}, \\
\partial^\alpha_x v_2 &= \frac{(v_2 - v_1)(v_2 - v_0)}{h},
\end{align*}
\] (17)

where

\[ h := ((v_2 - v_1)(v_0 - v_1) + K)(v_2 - v_0). \]

Note that \( h \) can be viewed as a constraint with the potential nonlinear system.

We may extend the aforementioned construction in Sections 2.1 and 2.2 into the complex \( z \)-plane. In this case, we utilize the Srivastava–Owa operator [16], which is defined as follows

\[ D^\lambda_z v(z) := \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{v(\xi)}{(z - \xi)^\lambda} d\xi; \quad 0 \leq \lambda < 1, \]

where the function \( v(z) \) is analytic in a simply connected region of the complex \( z \)-plane \( C \) containing the origin, and the multiplicity of \( (z - \xi)^{-\lambda} \) is removed by requiring \( \log(z - \xi) \) to be real when \( (z - \xi) > 0 \).

2.3. Matrix transform

In this section, our aim is to find a square root of stationary heat operators of arbitrary order. Let \( \mathcal{A} := C^\infty(S^1) \), \( \partial^\alpha_x = \frac{\partial}{\partial x^\alpha} \), \( \partial^\alpha_t = \frac{\partial}{\partial t^\alpha} \) and the fractional heat operator in \( (x, t) \) with potential \( \phi(x, t) \in \mathcal{A} \) and mass \( \mu \)

\[ \Delta = \partial^2_x + \phi - 2\mu \partial^\alpha_x, \]

and its conjugate

\[ \overline{\Delta} = \partial^2_x + \overline{\phi} - 2\mu \partial^\alpha_x. \]

By utilizing the properties of a matrix operators, we put

\[ P = \begin{pmatrix} \partial^\alpha_x & -2\mu \\ \partial^\alpha_t & -\partial^\alpha_x \end{pmatrix}, \]

and its conjugate

\[ \overline{P} = \begin{pmatrix} \partial^\alpha_x^* & -2\mu \\ \partial^\alpha_t^* & -\partial^\alpha_x^* \end{pmatrix}. \]

Here,

\[ (\partial^\alpha_x - \nu) \circ (\partial^\alpha_x + \nu) = \partial^2_x + \phi \]

is the Miura transform, in order to satisfy the fractional Riccati equation

\[ \phi = \partial^\alpha_x v - v^2 \]

and

\[ (\partial^\alpha_x + \nu) \circ (\partial^\alpha_x - \nu) = \partial^2_x + \overline{\phi} \]

in order to receive the fractional Riccati equation

\[ \overline{\phi} = -\partial^\alpha_x v - v^2. \]

Thus, we have the following

Theorem 2.1

Let \( (\partial^\alpha_x - \nu) \circ (\partial^\alpha_x + \nu) = \partial^2_x + \phi \) and \( (\partial^\alpha_x + \nu) \circ (\partial^\alpha_x - \nu) = \partial^2_x + \overline{\phi}. \) Then

\[ \frac{P^*\overline{P} + \overline{P}P}{2} = \left( \partial^2_x + v^2 - 2\mu \partial^\alpha_x \right) I_2. \]

For example, suppose that we have linear fractional differential equation of the form

\[ Ly = \partial^2_x y + \alpha_1 \partial^\alpha_x y + \alpha_0 = 0. \]

Then \( Ly \) may admit a factorization of the form

\[ L = (\partial^\alpha_x - \psi_2) \circ (\partial^\alpha_x - \psi_1). \]
where $\psi_1$ and $\psi_2$ satisfy the Riccati equations
\[
\frac{\partial}{\partial x} \psi_1 + \psi_1^2 + a_1 \psi_1 + a_0 = 0
\]
\[
\frac{\partial}{\partial x} \psi_2 - \psi_2^2 - a_1 \psi_2 - a_0 = 0.
\]
We may set
\[
D_1 = \begin{pmatrix}
\left( \frac{\partial}{\partial x} - \psi_2 \right) & a_1 \\
\frac{\partial}{\partial x} & -\left( \frac{\partial}{\partial x} - \psi_1 \right)
\end{pmatrix}
\]
and
\[
D_2 = \begin{pmatrix}
\left( \frac{\partial}{\partial x} - \psi_1 \right) & a_1 \\
\frac{\partial}{\partial x} & -\left( \frac{\partial}{\partial x} - \psi_2 \right)
\end{pmatrix}.
\]
Now, some simple computations imply
\[
D_1 D_2 = \begin{pmatrix}
\left( \frac{\partial}{\partial x} - \psi_2 \right) \circ \left( \frac{\partial}{\partial x} - \psi_1 \right) + a_1 \frac{\partial}{\partial x} & 0 \\
0 & \left( \frac{\partial}{\partial x} - \psi_2 \right) \circ \left( \frac{\partial}{\partial x} - \psi_1 \right) + a_1 \frac{\partial}{\partial x}
\end{pmatrix}
\]
\[
D_2 D_1 = \begin{pmatrix}
\left( \frac{\partial}{\partial x} - \psi_1 \right) \circ \left( \frac{\partial}{\partial x} - \psi_2 \right) + a_1 \frac{\partial}{\partial x} & 0 \\
0 & \left( \frac{\partial}{\partial x} - \psi_1 \right) \circ \left( \frac{\partial}{\partial x} - \psi_2 \right) + a_1 \frac{\partial}{\partial x}
\end{pmatrix}
\]
Hence, we have the following.

**Theorem 2.2**

Let
\[
\frac{\partial}{\partial x} \psi_1 + \psi_1^2 + a_1 \psi_1 + a_0 = 0,
\]
and
\[
\frac{\partial}{\partial x} \psi_2 - \psi_2^2 - a_1 \psi_2 - a_0 = 0.
\]
Then
\[
\frac{D_1 D_2 + D_2 D_1}{2} = \left( \frac{\partial}{\partial x} + a_1 \frac{\partial}{\partial x} + \psi_1 \psi_2 \right) I_2.
\]

By applying operators $P$ and $\overline{P}$, we may find the eigenvectors for heat operators as follows.

**Theorem 2.3**

If
\[
\begin{pmatrix}
\frac{\partial}{\partial x} + v & -2\mu \\
\frac{\partial}{\partial t} & \frac{\partial}{\partial x} + v
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix} = \lambda
\begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
\frac{\partial}{\partial x} - v & -2\mu \\
\frac{\partial}{\partial t} & \frac{\partial}{\partial x} - v
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix} = \kappa
\begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix}
\]
then $\Delta (\rho_1, \rho_2) = \lambda \kappa (\rho_1, \rho_2)$.

A similar argument can be used for the operators $D_1$ and $D_2$.

**Theorem 2.4**

If
\[
\begin{pmatrix}
\left( \frac{\partial}{\partial x} - \psi_2 \right) & a_1 \\
\frac{\partial}{\partial x} & -\left( \frac{\partial}{\partial x} - \psi_1 \right)
\end{pmatrix}
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix} = \mu
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
\left( \frac{\partial}{\partial x} - \psi_1 \right) & a_1 \\
\frac{\partial}{\partial x} & -\left( \frac{\partial}{\partial x} - \psi_2 \right)
\end{pmatrix}
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix} = \nu
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix},
\]
then $L (\sigma_1, \sigma_2) = \mu \nu (\sigma_1, \sigma_2)$. 

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