Fractional Alexander polynomials for image denoising

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ABSTRACT

Image denoising is an important task in image processing. The interest in using a fractional mask window operator based on fractional calculus has grown for image denoising. This paper mainly introduces the concept of fractional calculus and proposes a new mathematical method in using fractional Alexander polynomials for image denoising. The structures of $n \times n$ fractional mask windows on eight directions of this algorithm are constructed. Finally, we measure the denoising performance by employing experiments based on visual perception and by using peak signal-to-noise ratios. The experiments illustrate that the improvements achieved are compatible with other standard smoothing filters.

1. Introduction

Noise is any undesired signal that contaminates an image. Digital image acquisition is the primary process by which noise appears in digital images, converting an optical image into a continuous electrical signal. Noise, arising from a variety of sources, is inherent to all electronic image sensors and electronic components in the image environment. The goal of image denoising methods is to recover the original image contaminated by noise. Removing noise from the original signal remains an interesting topic for researchers. Several methods have been proposed to remove the noise and recover the true image, with each approach having its advantages and limitations [1].

Image denoising refers to the process of recovering a digital image that has been contaminated by all kinds of noise, while preserving as much as possible the textures and edges present in the image. Image denoising considered as an important task in image segmentation, feature extraction, and texture analysis. Traditionally, linear models, such as the Gaussian filter, have been commonly used to reduce noise. These methods perform well in the flat regions of images. However, their limitation is the inability to well-preserve the edges. The nonlinear model, however, can handle edges better than linear models. Another denoising method known as neighborhood filtering preserves a pixel by obtaining the average of the values of its neighbors [2]. Recently, many scholars have applied the theory of fractional calculus to image processing. Fractional calculus and its applications are important in several diverse areas of mathematical, physical, and engineering sciences. Fractional calculus generalizes ideas of the calculus of integrals and the derivatives of any arbitrary real or complex order. The advantages of fractional derivatives are obvious in modeling the mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids, rocks, and in many other fields [3,4].

Studies on fractional calculus that involve different operators, such as Riemann–Liouville, Erdélyi–Kober, Weyl–Riesz, Caputo, and Grünwald–Letnikov operators, have evolved during the past 40 years, and have extended in other fields. Fractional calculus in the field of image processing has gotten considerable attention in image
texture enhancement [5–7] and image denoising [8–11]. All the results that are based on fractional calculus operators showed that these methods are effective and reliable, and resulted in high levels of permanent immunity against different types of noise.

The fractional integral is extensively used in image denoising algorithms. Hu et al. [8, 12] proposed a fractional integral denoising algorithm and the implementation of a fractional integral filter using fractional integral mask windows on eight directions based on fractional calculus Riemann–Liouville definition. Simulation experiments showed the feasibility of the proposed fractional integral denoising algorithm. Guo et al. [13] proposed an image denoising algorithm using fractional integral mask windows based on the Grünwald–Letnikov definition of fractional calculus. Grünwald and Letnikov achieved fine-tuning of image denoising by setting a smaller fractional order and controlled the effect of image denoising by iteration.

In our previous study [9], we proposed a novel digital image denoising algorithm based on the generalized Srivastava–Owa fractional integral operator. The results illustrated that the proposed algorithm has a good upgrading of the denoised image. For image texture enhancement, Jalab and Ibrahim [6] proposed a texture enhancement technique using the fractional order Savitzky–Golay differentiator. This technique computes the generalized fractional order derivative of the input image using the sliding weight window over the image. Jalab and Ibrahim [6] proposed a texture enhancement technique for medical images using fractional differential mask windows based on the Srivastava–Owa fractional operators. Pu et al. [14] proposed fractional differential mask windows based on Grünwald–Letnikov and Riemann–Liouville for multiscale texture enhancement using six fractional differential masks. Experiments proved that the nonlinearly enhancing complex texture in a smooth area by fractional differential-based approach improves to be visibly better than that by traditional integral-based algorithms. Gao et al. [15] proposed image enhancement based on improved fractional differentiation by piecewise quadratic interpolation equation. Experiments showed that for texture-rich digital image, the capability of nonlinearly enhancing comprehensive texture details by improved fractional differentiation is obvious.

In this paper, we utilize the concept of fractional calculus for image denoising to generalize the Alexander polynomial in two approaches, namely, Alexander polynomial–fractional differential (AFD) and Alexander polynomial–fractional integral (AIF).

The Alexander polynomials advantage over the other techniques is that these polynomials can be computed by utilizing a skein relation, which can be employed in various topics in mathematics and physics, such as operator algebras and statistical mechanics [16].

The structures of \( n \times n \) fractional mask windows of these algorithms are constructed. The denoising performance is measured by employing experiments based on the visual perception and by using peak signal-to-noise ratio (PSNR). The remainder of this paper is organized as follows. In Section 2, we introduce the generalized fractional differential and fractional integral of the Alexander polynomial. The construction of the fractional differential mask windows, which is the new method proposed in this work, is presented in Section 3. The experimental results and the comparison with other studies are shown in Sections 4 and 5, respectively. Finally, the conclusion is presented in Section 6.

2. Alexander polynomial

The Alexander polynomial is a knot invariant created in 1923 by J.W. Alexander, with integer coefficients corresponding to each knot type. The Alexander polynomial was the only known knot polynomial until the Jones polynomial was derived in 1984. The Alexander polynomial is the main tool used to discuss a pair of curves known as a Zariski pair. This pair can be defined as follows: a couple of curves \( C_1 \) and \( C_2 \) of equal degree is used to design a Zariski pair. If neighborhoods exist, then \( T(C_i) \subset P^2 \) (projective plane) of \( C_i, \ i = 1, 2 \) such that \( (T(C_1), C_1) \) and \( (T(C_2), C_2) \) are diffeomorphic, while the pairs \( (P^2, C_1) \) and \( (P^2, C_2) \) are not homeomorphic (topologically not equivalent). Our aim is to construct two types of mask windows utilizing the Alexander polynomial and its generalization.

Definition 1. The Alexander polynomial is written as [17]

\[
\Delta(t) = \prod_{n=1}^{d-1} \Delta_n(t)^d, \quad n = 1, ..., d - 1
\]

where \( \varepsilon_n \) is a positive integer and

\[
\Delta_n(t) = \left( t - \exp \left( \frac{2\pi i n}{d} \right) \right) \left( t - \exp \left( -\frac{2\pi i n}{d} \right) \right).
\]

From (1), we can conclude the following \( \Delta_n(t) \):

\[
\Delta_1 = \Delta_{11} = t^2 - \sqrt{3}t + 1,
\]
\[
\Delta_2 = \Delta_{10} = t^2 - t + 1,
\]
\[
\Delta_3 = \Delta_9 = t^2 + 1,
\]
\[
\Delta_4 = \Delta_8 = t^2 + t + 1,
\]
\[
\Delta_5 = \Delta_7 = t^2 + \sqrt{3}t + 1,
\]
\[
\Delta_6 = (t + 1)^2.
\]

2.1. Fractional calculus

The idea of fractional calculus was proposed over 300 years ago. Abel, in 1823, investigated the generalized tautochrone problem and, for the first time, applied fractional calculus techniques in a physical problem. Liouville subsequently applied fractional calculus to problems in potential theory. Since that time, fractional calculus has captured the attention of many researchers in all areas of sciences [4].

This subsection deals with some preliminaries and notations regarding fractional calculus.

Definition 2. The fractional (arbitrary) order integral of the function \( f \) of order \( \alpha \) > 0 is defined by

\[
\mathcal{I}_a^f(\tau) = \int_a^\tau \left( \tau - \tau' \right)^{\alpha - 1} \frac{f(\tau') d\tau'}{\Gamma(\alpha)},
\]

when \( \alpha = 0 \), we write \( \mathcal{I}_a^f(\tau) = f(\tau) \ast \phi_\alpha(\tau) \), where \( (\ast) \) denoted the convolution product, \( \phi_\alpha(t) = (t^{\alpha - 1} / \Gamma(\alpha)) \), \( t > 0 \) and
\( \phi_o(t) = 0, \ t \leq 0 \) and \( \phi_o \to \delta(t) \) as \( \alpha \to 0 \) where \( \delta(t) \) is the delta function.

**Definition 3.** The fractional (arbitrary) order derivative of the function \( f \) of order \( 0 < \alpha \leq 1 \) is defined by

\[
D^\alpha_a f(t) = \frac{d}{dt} \int_a^t (t - \tau)^{a-1} f(\tau) d\tau = \frac{d}{dt} t^{1-a} f(t).
\]

**Remark 1.** From Definitions 2 and 3, \( \alpha = 0 \), we have

\[
D^\alpha t^n = \frac{\Gamma(n+1)}{\Gamma(n+1-a)} t^{n-a}, \quad \alpha > 1; \quad 0 < \alpha < 1
\]

and

\[
F t^n = \frac{\Gamma(n+1)}{\Gamma(n+1+a)} t^{n+a}, \quad \alpha > 1; \quad \alpha > 0.
\]

We proceed to generalize the Alexander polynomial utilizing the Mittag-Leffler function

\[
E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(an+1)}
\]

Thus we obtain

\[
\Delta(t) = \prod_{n=1}^{d-1} \Delta_n(t)^{\epsilon_n}, \quad n = 1, ..., d-1
\]

where \( \epsilon_n \) is positive integer and

\[
\Delta_n(t) = \left( t - \frac{2n\pi i}{d} \right) \left( t - \frac{2n\pi i}{d} \right)
\]

**3. Construction of Fractional Mask Windows**

By using Remark 1 on (1), we receive two sets of fractional coefficients; the AFD set for \( t \neq 0 \):

\[
\Delta_1^a = \Delta_1^a = \frac{2}{\Gamma(3-a)} t^{2-a} - \frac{\sqrt{3}}{\Gamma(2-a)} t^{1-a} + \frac{t^a}{\Gamma(1-a)}
\]

\[
\Delta_2^a = \Delta_2^a = \frac{2}{\Gamma(3-a)} t^{2-a} - \frac{1}{\Gamma(2-a)} t^{1-a} + \frac{t^a}{\Gamma(1-a)}
\]

\[
\Delta_3^a = \Delta_3^a = \frac{2}{\Gamma(3-a)} t^{2-a} + \frac{t^a}{\Gamma(1-a)}
\]

\[
\Delta_4^a = \Delta_4^a = \frac{2}{\Gamma(3-a)} t^{2-a} + \frac{\sqrt{3}}{\Gamma(2-a)} t^{1-a} + \frac{t^a}{\Gamma(1-a)}
\]

\[
\Delta_5^a = \Delta_5^a = \frac{2}{\Gamma(3-a)} t^{2-a} + \frac{2}{\Gamma(2-a)} t^{1-a} + \frac{t^a}{\Gamma(1-a)}
\]

And the FDI set

\[
\Delta_1^a = \Delta_1^a = \frac{2}{\Gamma(3-a)} t^{2+a} - \frac{\sqrt{3}}{\Gamma(2-a)} t^{1+a} + \frac{t^a}{\Gamma(1+a)}
\]

\[
\Delta_2^a = \Delta_2^a = \frac{2}{\Gamma(3-a)} t^{2+a} - \frac{1}{\Gamma(2-a)} t^{1+a} + \frac{t^a}{\Gamma(1+a)}
\]

\[
\Delta_3^a = \Delta_3^a = \frac{2}{\Gamma(3-a)} t^{2+a} + \frac{t^a}{\Gamma(1+a)}
\]

\[
\Delta_4^a = \Delta_4^a = \frac{2}{\Gamma(3-a)} t^{2+a} + \frac{1}{\Gamma(2-a)} t^{1+a} + \frac{t^a}{\Gamma(1+a)}
\]

\[
\Delta_5^a = \Delta_5^a = \frac{2}{\Gamma(3-a)} t^{2+a} + \frac{\sqrt{3}}{\Gamma(2-a)} t^{1+a} + \frac{t^a}{\Gamma(1+a)}
\]

\[
\Delta_6^a = \frac{2}{\Gamma(3-a)} t^{2+a} + \frac{2}{\Gamma(2-a)} t^{1+a} + \frac{t^a}{\Gamma(1+a)}
\]

\[
\Delta_7^a = \frac{2}{\Gamma(3-a)} t^{2+a} + \frac{\sqrt{3}}{\Gamma(2-a)} t^{1+a} + \frac{t^a}{\Gamma(1+a)}
\]

\[
\Delta_8^a = \frac{2}{\Gamma(3-a)} t^{2+a} + \frac{2}{\Gamma(2-a)} t^{1+a} + \frac{t^a}{\Gamma(1+a)}
\]

\[
\Delta_9^a = \frac{2}{\Gamma(3-a)} t^{2+a} + \frac{2}{\Gamma(2-a)} t^{1+a} + \frac{t^a}{\Gamma(1+a)}
\]

\[
\Delta_{10}^a = \frac{2}{\Gamma(3-a)} t^{2+a} + \frac{2}{\Gamma(2-a)} t^{1+a} + \frac{t^a}{\Gamma(1+a)}
\]

where \( n = 1, 2, ..., 9 \) represents the location of pixel inside each mask window.

The final new filtered image based on either AFD or AFI can be obtained by the summation of all eight convolution results of the magnitudes for each filter.

In this paper, filter scheme of (AFD, and AFI) approaches are based on direct processing for discrete pixels, by moving the mask window pixel by pixel. We implement the fractional mask windows on the eight symmetric directions using AFD operators, which has the same structure as AFI but with different coefficients. They should have the functions that the result of the convoluting will remove the noise from the corrupted images, which is the main aim of this study.

Algorithmic complexity is concerned about how fast or slow a particular algorithm performs. The goal of computational complexity is to classify the proposed algorithm according to its performances. We use the "big-O" notation to express an algorithm runtime complexity. Big-O notation is
a mathematical construct used to describe algorithmic complexity. The technical viewpoint of this work is based on splitting the entire image into sub-image blocks.

In this paper, if we consider each image block having a size of \((n \times n)\), then the processing time of each fractional mask window on each block for all images would be

![Figure 1](image_url)

**Fig. 1.** Fractional differential mask windows on eight directions: 180°, 90°, 0°, 270°, 45°, 135°, 315°, and 225°.
\[ T_p = 8n^2, \]  
\[ T_c \]  
\[ T_f = O(8n^2T_c) \]  
(7)

Which indicates that the complexity of the denoising algorithms depends mainly on the mask window size \((n \times n)\), and to achieve higher denoising with lower complexity, the size of the mask window should be small.

The steps for the proposed fractional mask window image denoising algorithm can be given as follows:

1. Resize the images to 512 \times 512 pixels.
2. Initialize fractional differential and fractional integral windows of 3 \times 3 size.
3. Define the values of the fractional powers of the proposed mask windows with the range of \(0 < \alpha \leq 1\) and \(0 < t < 1\).
4. Add artificial noise to the input image (Gaussian, speckle, and Rician noise).
5. Apply the proposed AFD, and AFI filters.
6. Apply different smoothing filters (Gaussian filter, Homomorphic Wiener filter, and Kuan filter).
7. Calculate the PSNR between the corrupted image and the denoised image.

The same algorithm, which is used for grayscale images, can be applied for color images, but is performed separately for each of the red, green, and blue color components.

4. Experimental results and discussion

Performance tests for the system proposed in this paper were implemented using Matlab 2013b on system type 64-bit Windows 7.

Fig. 3. Choice of parameter \(t\) for grayscale images corrupted by Gaussian noise with standard deviation \(\sigma\) value of 25. (a) AFD algorithm and (b) AFI algorithm.
The four sets of images employed here are as follows:

1. Grayscale images “Lena”, “Cameraman”, and “Boat”.
3. Ultrasonic image.
4. MRI image.

We study the performance of the proposed approach using images corrupted by Gaussian speckle, and Rician noise. Both proposed AFD, and AFI filters are considered to operate using $3 \times 3$ processing mask windows. The performance of the proposed algorithm was evaluated by computing the PSNR.

**Fig. 4.** Experiment with artificial Gaussian noise. Grayscale images “Lena”. (a) Original image, (b) image with Gaussian noise with $\sigma=15$, (c) Gaussian smoothing filter, (d) homomorphic Wiener filter $3 \times 3$, (e) proposed AFD filter, and (f) AFI proposed filter.
The PSNR is defined via the mean squared error (MSE) for two images, namely, \( I \) and \( K \), where one of the images is considered the corrupted image and the other is the denoised image respectively.

\[
MSE = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} [(I(i,j) - K(i,j))^2]
\]  

(8)

\[
PSNR = 10 \log_{10} \frac{\text{max}(I, K)^2}{MSE}
\]  

(9)

where \( \text{max} \) is the maximum possible pixel value of the image, \( \text{max} \) is equal to 255 in a grayscale image.

![Fig. 5. Experiment with artificial Gaussian noise. Color images “Peppers”. (a) Original image, (b) image with Gaussian noise with \( \sigma = 15 \), (c) Gaussian smoothing filter, (d) homomorphic Wiener filter \( 3 \times 3 \), (e) proposed AFD filter, and (f) AFI proposed filter. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)](image-url)
4.1. Selection of fractional power parameters

The two fractional power parameters in our algorithm are $\alpha$ and $t$. In Fig. 2, we first display the behavior of PSNR for the values of $\alpha$, ranging from 0.1 to 1, when applying the proposed algorithm on the three different images corrupted by Gaussian noise.

A small $\alpha$ value leads to a small value of the PSNR of the denoised image, whereas a large $\alpha$ value leads to a dramatic decrease of the PSNR. Therefore, the trade-off

Fig. 6. Experiment with artificial Gaussian noise. Grayscale images “Boat”. (a) Original image, (b) image with Gaussian noise with $\sigma=20$, (c) Gaussian smoothing filter, (d) homomorphic Wiener filter $3 \times 3$, (e) proposed AFD filter, and (f) AFI proposed filter.
between $\alpha$ and PSNR is required to remove the noise. We should choose the optimal value of $\alpha$, which is shown in Fig. 2. We choose $\alpha=0.88$ for both AFD and AFI algorithms. Fig. 3 shows the behavior of the PSNR for values of $t$, varying from 0.1 to 1, when applying our algorithm on the three images corrupted by Gaussian noise. The trade-off between $t$ and PSNR is chosen as the optimal value of $t$, which is shown Fig. 3. We choose $t=0.58$ for the AFD algorithm and $t=0.88$ for the AFI algorithm.

4.2. AFD and AFI

For the human visual system effect, we make three experimental tests as follows:

The first test consists of three grayscale images and three colored images, all are corrupted by Gaussian noise $\sigma=15, 20, \text{and } 25$.

The second test deals with the ultrasonic image when it is corrupted by Speckle noise with variance $=0.04$. The last is the MRI image corrupted by Rician noise with $\sigma_c=10$.

Fig. 7. Experiment with artificial Gaussian noise. Color images "Baboon": (a) Original image, (b) image with Gaussian noise with $\sigma=20$, (c) Gaussian smoothing filter, (d) homomorphic Wiener filter $3 \times 3$, (e) proposed AFD filter, and (f) AFI proposed filter. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
The experimental results of all images are shown in Figs. 4–11. Figs. 4–9 show the results of denoising of different images corrupted with different artificial Gaussian noise \( \sigma \). While Fig. 11 shows the ultrasonic image with artificial noise \( v=0.04 \), and MRI image with artificial Rician noise \( \sigma_c=10 \).

Figs. 4–11 show that the proposed AFD and AFI algorithms have good denoising performance for all testing images.

The aim of this study is to ensure that our proposed algorithm removes noise effectively from the corrupted images by comparing our algorithm with three standards.
filters for image denoising, which are the standard Gaussian filter, Kuan Filter, and Homomorphic Wiener filter.

Table 1 shows the results of PSNR obtained by different filters for two sets of standard images (grayscale and color images) with 512 x 512 pixels. The images were corrupted with Gaussian noise with different values of $\sigma$. Gaussian smoothing filter, homomorphic Wiener filter 3 x 3, proposed AFD filter, and AFI proposed filter were then applied to the corrupted images to remove the noise. The performance of the denoising process was quantified using PSNR. The maximum PSNR value was obtained by our proposed algorithms using the

Fig. 9. Experiment with artificial Gaussian noise. Color images “House”. (a) Original image, (b) image with Gaussian noise with $\sigma=25$, (c) Gaussian smoothing filter, (d) homomorphic Wiener filter 3 x 3, (e) proposed AFD filter, and (f) AFI proposed filter. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
optimal values of $\alpha$ and $t$. For the corrupted images by Gaussian noise, our proposed algorithm shows better results than standard Gaussian filter, and Homomorphic Wiener filter.

Although the PSNR rate is slightly higher than Gaussian filter and Homomorphic Wiener filter. The reason is due to the fact that the enhancing of the fractional operators only affects the pixel values that are changing sharply (high frequency of image), while no significant changes occur in the low frequency of image [14].

Table 2 shows the result of PSNR obtained for the ultrasonic image corrupted by the artificial Speckle noise, and the MRI image corrupted by artificial Rician noise. The Kuan filter, Homomorphic Wiener filter, and proposed AFD and AFI were
applied to the corrupted images. The maximum PSNR value was obtained by our proposed algorithm. From the experiment results, it can be seen that our proposed algorithm achieves some significant improvements over Kuan, and Homomorphic Wiener filters for Speckle and Rician noise.

5. Quantitative comparison with other methods

The visual and quantitative measures of proposed algorithm are compatible to several existing filters for image denoising. A comparative analysis of our study with
other methods is shown in Tables 3 and 4. To our knowledge, no previous studies have been done using fractional differential for image denoising. Therefore, we compare our study with those which employed other methods. Table 3 shows the comparison of the experimental results of the proposed algorithm with other denoising algorithms for “Boat” image with the noise standard deviation \( \sigma \) values of 15, 20, and 25. Hu et al. [8] proposed a novel image denoising algorithm named fractional integral image denoising algorithm, which is based on the fractional calculus Riemann–Liouville definition, while Jalab and Ibrahim [9] proposed an image denoising algorithm called generalized fractional integral filter based on the generalized Srivastava–Owa fractional integral operator.

Tables 3 and 4 provide an overall view of the performance of different methods, although these methods have used different images with different noise standard deviation \( \sigma \) values. As shown in these tables, for both testing images “Boat” and “Lena”, the values of PSNR for the proposed algorithm (AFD and AFI) are slightly larger than the three standard methods for the noise standard deviation \( \sigma \) values of 15 and 20 and are better than the existing filters when the image is highly corrupted by Gaussian noise.

The proposed algorithm for the image denoising algorithm provides satisfactory results. The good PSNR of the proposed algorithm acts as one of the important parameters in judging its performance.

6. Conclusion

An image denoising algorithm based on fractional Alexander polynomials is introduced. The structures of \( n \times n \) fractional mask windows of this algorithm are constructed. The denoising performance is measured by employing experiments based on visual perception and PSNR values. Experiments demonstrate that the improvements achieved are compatible with the standard Gaussian smoothing, Kuan, and Homomorphic Wiener filters. An additional interesting property of our proposed algorithms is the characteristic of the denoising filter that can be adjusted easily by changing the two values of the fractional powers of the proposed mask windows. Future studies must be conducted for texture enhancement of digital images using both AFD and AFI algorithms.

Authors’ contributions

All authors contributed equally and significantly in writing this paper and approved the final manuscript.

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