Friedmann Equation with Quantum Potential

Ch’ng Han Siong, Shahidan Radiman and Bijan Nikouravan

Abstract. Friedmann equations are used to describe the evolution of the universe. Solving Friedmann equations for the scale factor indicates that the universe starts from an initial singularity where all the physical laws break down. However, the Friedmann equations are well describing the late-time or large scale universe. Hence now, many physicists try to find an alternative theory to avoid this initial singularity. In this paper, we generate a version of first Friedmann equation which is added with an additional term. This additional term contains the quantum potential energy which is believed to play an important role at small scale. However, it will gradually become negligible when the universe evolves to large scale.

Keywords: Friedmann equations, de Broglie-Bohm interpretation, quantum potential.

PACS: 04.20.Fy, 04.60.Ds, 98.80.Qc

INTRODUCTION

Albert Einstein had developed the theory of general relativity in 1915. The central equation in this theory is the Einstein’s field equation. Einstein’s field equation gives the connection between geometry of space-time and the source of energy-momentum. The Einstein’s field equation is given as follows [1]:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \]  \hspace{1cm} (1)

where \( R_{\mu\nu} \), \( R \), \( g_{\mu\nu} \), \( T_{\mu\nu} \), \( G \) and \( c \) are Ricci tensor, Ricci scalar, metric tensor, energy-momentum tensor, Newton’s gravitational constant and the speed of light respectively. The left hand side of Einstein’s field equation gives a type of geometry of space-time which is governed by the type of energy-momentum on the right hand side of equation.

From the Cosmological Principle, the universe is taken to be homogeneous and isotropic. Hence, the metric of the universe has the following form:

\[ ds^2 = -c^2 dt^2 + a^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \]  \hspace{1cm} (2)

where \( a = a(t) \) and \( k \) are the scale factor and curvature parameter respectively. The curvature parameter, \( k \) takes one of the three values, \(-1\), \(0\) or \(+1\) for the spatially open, flat or closed universe respectively. Here, the coordinate \( r \) is unit less, while the scale factor \( a \) takes the unit of length. The metric (2) is called the Robertson-Walker metric. To know how the scale factor evolves with time, we need to plug in the Robertson-Walker metric into Einstein’s field equation.

In 1922, Alexander Friedmann solved the Einstein’s field equation for the Robertson-Walker metric. The two Friedmann equations are as follows:
\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{c^2 k}{a^2} = \frac{8\pi G \varepsilon}{3c^2} \quad (3)
\]

and

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\varepsilon + 3p) \quad (4)
\]

Equation (3) is also called the first Friedmann equation. The symbol \( \dot{\varepsilon} \) and \( p \) denote the differentiation with respect to time, energy density and pressure respectively. The energy density and pressure can also be written as follows [2]:

\[\varepsilon = \frac{n_{(n)}}{a''}\]

and

\[p = w\varepsilon\]

where \( n_{(n)} \) is a constant. For different type of mass-energy, the value of \( n \) as well as \( w \) would be different. We summarize it in the following table:

<table>
<thead>
<tr>
<th>Type of Mass-Energy</th>
<th>( w )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radiation</td>
<td>( 1/3 )</td>
<td>4</td>
</tr>
<tr>
<td>Pressure Less Matter</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Vacuum Energy</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

In this paper, we derive a new version of first Friedmann equation from de Broglie-Bohm interpretation of the wave function of the universe. This new version of first Friedmann equation has an additional term which contains the quantum potential energy. Now, we call this new version of first Friedmann equation as first quantum Friedmann equation. De Broglie-Bohm interpretation [3-4] is an alternative to the orthodox Copenhagen interpretation. In this interpretation, the particle and field are not only acted upon by classical potential energy but also by quantum potential energy. Quantum potential energy is responsible for all the quantum effects.

In section 2, we present the spatially flat cosmological model. In section 3, we derive the first quantum Friedmann equation. Finally, we have some discussions in section 4.

**A SPATIALLY FLAT COSMOLOGICAL MODEL**

We start by introducing the Einstein-Hilbert action in a Cartesian coordinate [5-6]:

\[
S = \int \sqrt{-g} \left[ \frac{R c^4}{16\pi G} - \varepsilon \right] d\chi d\eta d\zeta d\tau \quad (7)
\]

The Einstein’s field equation is obtained when we take to vary the above action based on the principle of least action. Choosing \( k = 0 \), the metric (2) in Cartesian coordinate would become
\[ ds^2 = -c^2 dt^2 + a^2 \left( dx^2 + dy^2 + dz^2 \right) \]  

(8)

Splitting this form of space-time into time and space variables, the metric (8) of a spatially flat universe becomes

\[ ds^2 = -N^2 c^2 dt^2 + a^2 \left( dx^2 + dy^2 + dz^2 \right) \]  

(9)

where \( N = N(t) \) is the lapse function. Subsequently, the Ricci scalar and determinant of metric (9) are computed and given as follows:

\[
R = \frac{6\ddot{a}}{N^2 c^2 a} - \frac{6\dot{N}\dot{a}}{N^3 c^2 a} + \frac{6\dot{a}^2}{N^2 c^2 a^2}
\]

(10)

\[
g = -N^2 a^6
\]

(11)

Next, we substitute the Ricci scalar (10) and determinant of metric (9) into Einstein-Hilbert action (7). Hence, we have

\[
S = \int \left[ \frac{c^2}{16\pi G} \left( \frac{6\ddot{a}^2}{N} - \frac{6\dot{N}\dot{a}^2}{N^2} + \frac{6\dot{a}^2}{N} \right) - Na^3 \epsilon \right] dx\, dy\, dz\, c\, dt
\]

(12)

After integrating the first term by parts with respect to \( t \), equation (12) becomes

\[
S = V_o \int \left[ -\frac{3a\ddot{a}^2 c^2}{8\pi GN} - Na^3 \epsilon \right] dt
\]

(13)

where \( V_o = c \int dx\, dy\, dz \). We can set \( V_o \) equal to one. This is reasonable because for any given value of \( t \), the geometry of our universe is the same everywhere. Hence, \( V_o = c \int dx\, dy\, dz \) is constant and can be set equal to one by integrating over an appropriate compact region of space [7]. We take the assumption that the energy density \( \epsilon \) in (13) is a type of mass-energy that dominates the universe. Hence, we can substitute (5) into (13). The Einstein-Hilbert action (13) becomes

\[
S = \int \left[ \frac{3a\ddot{a}^2 c^2}{8\pi GN} - \frac{Na^3 \lambda_{(n)}}{a^n} \right] dt
\]

(14)

The Lagrangian \( L \) from Einstein-Hilbert action (14) is given as follows:

\[
L = -\frac{3a\ddot{a}^2 c^2}{8\pi GN} - \frac{Na^3 \lambda_{(n)}}{a^n}
\]

(15)

Subsequently, the momentum conjugate to \( a \) is computed and given as
\[ P_a = \frac{\partial L}{\partial \dot{a}} = -\frac{3a\dot{a}c^2}{4\pi GN} \]

Hence, the Hamiltonian is given by

\[ H = P_a \dot{a} - L = N \left( -\frac{2\pi GP_a^2}{3ac^2} + \frac{a^3 \lambda(n)}{a^n} \right) \]

The Hamiltonian constraint (for details, see [8]) provides \( H = 0 \). We thus have the following equation:

\[ -\frac{2\pi GP_a^2}{3ac^2} + \frac{a^3 \lambda(n)}{a^n} = 0 \]

We next apply the canonical quantization procedure to equation (18). Replacing \( P_a \) by \(-i\hbar \frac{d}{da}\), where \( i \) is an complex number and \( \hbar \) is the reduced Planck constant and imposing \( \hat{H}\psi = 0 \), we obtain the following Wheeler-DeWitt equation:

\[ \left[ \frac{2\pi G\hbar^2}{3ac^2} \frac{d^2}{da^2} + \lambda(n) a^{3-n} \right] \psi = 0 \]

where \( \psi \) is the wave function of the universe.

**DERIVATION OF QUANTUM FRIEDMANN EQUATION**

The wave function of the universe \( \psi \) is a complex number, hence we may write it as the following form:

\[ \psi = R \cdot \exp \left[ \frac{iS}{\hbar} \right] \]

where \( R \) and \( S \) are functions of the scale factor \( a \). To illustrate the de Broglie-Bohm interpretation, we substitute the wave function (20) into the Wheeler-DeWitt equation (19). After taking the derivatives and separating into the real and imaginary parts, we have the following two equations:

\[ -\frac{2\pi G}{3c^2a} (\nabla S)^2 + a^3 \frac{\lambda(n)}{a^n} + \frac{2\pi G\hbar^2}{3c^2a} \nabla^2 R = 0 \]

and

\[ \nabla \left( R^2 \nabla S \right) = 0 \]

The symbol \( \nabla \) denotes the differentiation with respect to the scale factor \( a \). Equation (22) is known as the continuity equation for probability, while equation (21) is treated as the modified Hamilton-Jacobi equation. It differs from the usual classical Hamilton-Jacobi equation by an additional term which is as follows:
\[ Q = \frac{2\pi G\hbar^2}{3c^2a} \frac{\nabla^2 R}{R} \]  

(23)

The term (23) is called the quantum potential energy. This is the vital element in the de Broglie-Bohm interpretation. Let us now make the following substitutions [9]:

\[ m' = -\frac{3c^2a}{4\pi G} \]

(24)

and

\[ V = a^3 \frac{\lambda^{(n)}}{a^n} \]

(25)

Hence, equation (21) would become [9]

\[ \left( \frac{\nabla S}{2m'} \right)^2 + V - \frac{\hbar^2}{2m'} \frac{\nabla^2 R}{R} = 0 \]

(26)

Equation (26) is analogous to the modified Hamilton-Jacobi equation for a particle with Hamiltonian \( H = 0 \). Therefore, it is reasonable to have the following relation \( \nabla S = P_a \) for equation (21). Replacing \( \nabla S \) in equation (21) by \( P_a \) (16) and taking a gauge choice of \( N = 1 \), we obtain the following equation:

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G\varepsilon}{3c^2} + \frac{8\pi GQ}{3c^2a^3} \]

(27)

Equation (27) is very similar to the spatially flat \( (k = 0) \) Friedmann equation (3), except there is an additional term on the right hand side of the equation. This is the first quantum Friedmann equation. Let us now take \( T_1 = \frac{8\pi GQ}{3c^2a^3} \) for this additional term. Hence, equation (27) can be written as follows:

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G\varepsilon}{3c^2} + T_1 \]

(28)

To obtain the second quantum Friedmann equation, we differentiate both sides of equation (28) with respect to time \( t \) to obtain the following equation:

\[ \frac{\ddot{a}}{a} = \left( \frac{\dot{a}}{a} \right)^2 - \frac{4\pi G\varepsilon}{3c^2} \frac{\lambda^{(n)}}{a^n} + \frac{a^2\dot{T}_1}{2a} \]

(29)

From the TABLE (1), we could write down an equation relating \( n \) and \( w \):

\[ n = 3(1 + w) \]

(30)
Substituting equation (30) into (29), we obtain

$$\frac{\ddot{a}}{a} = \left(\frac{\dot{a}}{a}\right)^2 - \frac{12\pi G(1 + w)\hat{n}}{3c^2}a^n + \frac{a\dot{\hat{n}}}{2\dot{a}}$$

(31)

We now substitute equation (28) into (31). Equation (31) becomes

$$\frac{\ddot{a}}{a} = \frac{8\pi Ge}{3c^2} + T_1 = \frac{12\pi G\hat{n}}{3c^2} - \frac{12\pi G\hat{n}}{3c^2}a^n + \frac{a\dot{\hat{n}}}{2\dot{a}}$$

(32)

Substituting equations (5) and (6) into (32), we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3p) + \frac{a\dot{\hat{n}}}{2\dot{a}} + T_1$$

(33)

Equation (33) is the second quantum Friedmann equation. Let us now take $T_2 = \frac{a\dot{\hat{n}}}{2\dot{a}} + T_1$. Hence, equation (33) can be written as follows:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3p) + T_2$$

(34)

**DISCUSSIONS**

We have derived the first and second quantum Friedmann equations. The difference between these quantum Friedmann equations and the classical Friedmann equations (3) and (4) is the additional extra terms: $T_1$ and $T_2$. These extra terms $T_1$ and $T_2$ are expected to play the role for quantum effects especially when the universe starts to begin. However, these extra terms are also expected to be gradually negligible when the scale of the universe is increasing. The classical Friedmann equations yield the initial singularity for the universe at which all the physical laws break down. We thus expect that the extra terms $T_1$ and $T_2$ in quantum Friedmann equations would avoid this singularity problem. We are difficult to obtain the “true” wave function of the universe. Physicists now try to compute the wave function of the universe which is based on some assumed initial conditions. However, we do not know the exact initial condition of the universe.

**ACKNOWLEDGMENTS**

One of the authors, Ch’ng would like to thank the Ministry of Higher Education (MOHE), Malaysia for providing MY PhD scholarship.

**REFERENCES**


