Beam Splitter Entangler for Nonlinear Bosonic Fields

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Abstract—Some years ago Katriel and Solomon [1] described applications to the characterization of the photon statistics of nonideal lasers, nonclassical light, and deformed photon states using $f$-deformed coherent states. In this letter, we study the effect of a beam splitter on these nonlinear coherent states. We find that these states are useful for generating quantum entanglement as the deformation parameter gets farther from the unity and for strong input field regimes. The results are confirmed through the Werhl entropy.

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INTRODUCTION

Entanglement is a fascinating property of quantum systems, that has various important consequences for modern physics applications. It has been the focus of foundational discussions of quantum mechanics since the time of Schrödinger and the paper of Einstein et al. [2, 3]. Entanglement lies at the heart of many problems in quantum mechanics and has attracted an increasing attention in recent years [4–7]. It is needed in several quantum information processing tasks such as teleportation and certain quantum cryptographic protocols. It also plays an important role in quantum computing making it possible that quantum computers can outperform their classical counterparts for several problems such as prime factoring or searching. Furthermore, the generation of quantum entanglement naturally arises as goals in nowadays quantum control experiments when studying the nonclassical phenomena in quantum mechanics. When in an experiment entanglement is created, it is important to detect it. Thus, in many quantum physics experiments the creation of an entangled state is followed by measurements. Based on the results of these measurements, the experimenters conclude that the produced state was entangled. However, in many-particle experiments the possibilities for quantum control are very limited. In particular, the particles cannot be individually addressed. In such systems, the entanglement can be created and detected with collective operations.

Recently, various devices have been proposed and realized experimentally to generate quantum entanglement, such as beam splitter [8, 9], Cavity QED [10], NMR systems [11], etc. The entanglement generated by a beam splitter has been studied in considerable detail by Kim et al. [12]. They showed that, in order to obtain entangled output states of a beam splitter, a necessary condition is that at least one of the input fields should be nonclassical. It was recently shown by Ivan et al. that if the input state to a beam splitter is a product state of the vacuum in one mode and a state with a sub-Poissonian distribution, then the output state will be entangled [13]. More recently, this result is generalized to the states with super-Poissonian and sub-Poissonian statistics, by investigating the statistical properties of generalized spin coherent states for different kinds of deformations [14].

Another concept widely used and applied in quantum information theory is the notion of coherent or quasiclassical states. These states make a very useful tool for the investigation of various problems in physics. Coherent states were first introduced by Schrödinger [15] in the context of the harmonic oscillator, who was interesting in finding quantum states which provide a close connection between quantum and classical formulations of a given physical system. Later, the notion of coherent has became very important in quantum optics due to Glauber [16], as eigenstates of the annihilation operator a of the harmonic oscillator, while he demonstrated that these states have the interesting property of minimizing the Heisenberg uncertainty relation. On the other hand, the quantum groups were introduced as a mathematical description of deformed Lie algebra that gave the possibility to construct deformed coherent states. They were introduced as a natural extension of the notion of coherent states [17, 18]. Recently, these states have attracted a
lot of attention due to their possible applications in various branches of physics [19–22]. Such states exhibit some nonclassical properties such as photon antibunching [23], sub-Poissonian photon statistics [24] and squeezing [25, 26], etc. (for a review see [18]). Moreover, it has been experimentally observed that the real laser, bunched and antibunched light possess a photon number statistics which can be super-Poissonian or sub-Poissonian [27, 28].

Here, we are going to study the effect of a beam splitter on the $f$-deformed coherent states and analyze the entanglement generated in terms of the parameters involved in the coherent states. We study the Wehrl entropy of the input optical field and explore a link between this quantity and the output state entanglement. In particular, we shall show that the Wehrl entropy may be used as an indicator of the entanglement behavior in this scheme.

DEFORMED BOSONIC COHERENT STATES

We begin our discussion by presenting the overall state of the art of the $f$-deformed bosonic coherent states and their properties.

In this letter, we are interested to explain $f$-deformed quantum harmonic oscillator algebra ($f$-deformed Heisenberg–Weyl algebra) defined by the algebra generated by the operator \{1, $A$, $A^\dagger$, N\}. The $f$-deformed oscillator operators are defined as follows

\[
A = af(N) = f(N + 1) a, \\
A^\dagger = f(N) a^\dagger = a^\dagger f(N + 1),
\]

(1)

where $N = a a^\dagger$ is the usual number operator and $a$, $a^\dagger$ are the annihilation and creation boson operators of the ordinary harmonic oscillator algebra respectively. They obey the following commutation relations

\[
[N, A] = -A, \quad [N, A^\dagger] = A^\dagger, \\
[A, A^\dagger] = (N + 1) f^2(N + 1) - N f^2(N).
\]

(2)

The function $f$, which is a characteristics for the deformation, has a dependance on a deformation parameter $q$ where in the $q \to 1$ limit ($f(q = 1)$), the deformation disappears and the usual quantum algebra is recovered.

Recently, Mancini and Man’ko [29] have studied the dissipative dynamics of $f$-deformed coherent states superposition. They have found that such a kind of superposition can be more robust against decoherence than the usual Schrödinger cat states. The coherent states of $f$-deformed quantum algebra defined as eigenstates of the $f$-deformed bosonic field $A$ [1, 29]

\[
|\alpha, f\rangle = N \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!} f(n)!} |n\rangle, \quad N = |\exp[f(\alpha^2)]|^{-\frac{1}{2}},
\]

(3)

where we have considered $\alpha \in \mathbb{R}$ and we have introduced

\[
\exp[x] = \sum_{n=0}^{\infty} \frac{x^n}{n! f(n)!}, \\
f(n)! = f(n)f(n-1)\ldots f(0).
\]

(4)

The function $\exp_x$ is a deformation version of the usual exponential function. They become coincident when $f$ is the identity. Notice that $\exp_x[xy] \neq \exp_x[x + y]$ and $\exp_x[x]^a \neq \exp_x[a x]$, i.e., we have a non-extensive exponential which can be found in many physical problems [30, 31].

EFFECT OF A BEAM SPLITTER ON $f$-DEFORMED COHERENT STATES

We first discuss how a beam splitter acts on an input state comprised of a state, $|\alpha, f\rangle$ to be studied, in one input port and a vacuum state $|0\rangle$ in the other port.

Quantum mechanically, the action of a beam splitter can be described by a unitary operator $\hat{U}_{bs}$ that relates the input state to the output state by

\[
|\text{out}\rangle = \hat{U}_{bs} |\text{int}\rangle
\]

(5)

where the beam splitter operator is [12]

\[
\hat{U}_{bs} = \exp \left[ \frac{i}{2} \left( a b^\dagger e^{i\phi} - a^\dagger b e^{-i\phi} \right) \right]
\]

(6)

here $a$ and $b$ are the annihilation operators of the two input fields and $a^\dagger$ and $b^\dagger$ are their Hermitian conjugates, and $\phi$ denotes the phase difference between the reflected and transmitted fields.

In order to obtain the beam splitter transformation when deformed bosonic fields are incident on one input port, we first introduce the effect of a beam splitter on an input state comprised of a Fock state in the input beam, that is $|n\rangle$, and a ground Fock state in the other

\[
\hat{U}_{bs} |n\rangle |0\rangle = \sum_{p=0}^{n} \left( \frac{n}{p} \right)^{\frac{1}{2}} T^p R^{(n-p)} |p\rangle |n-p\rangle.
\]

(7)

The quantities $T$ and $R$ are the transmissivity and the reflectivity of the beam splitter, respectively, obeying the normalization condition $|T|^2 + |R|^2 = 1$. Through the balance of this Letter, we shall assume that our beam splitter is 50:50 and that the reflected beam...
It is well known that if a Glauber state defined as
\[
|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle
\]
is incident on one input port and a vacuum state on the other, then the output state is separable with zero entanglement
\[
|\text{out}\rangle = \hat{U}_{bs}|\alpha\rangle|0\rangle = \left|\frac{\alpha}{\sqrt{2}}\right\rangle \otimes \left|\frac{i\alpha}{\sqrt{2}}\right\rangle.
\]
The Glauber coherent states are the only states that when passed through one input port of a beam splitter result in product states at the output. Any other states at the input result in entangled states at the output [32].

Now let us examine the entanglement generated via 50:50 beam splitter when a \(f\)-deformed coherent state is injected into one input port and a vacuum state is injected into the other. In this case, the output state is obtained by using both Eqs. (5) and (7)

\[
|\text{out}\rangle = \hat{U}_{bs}|\alpha\rangle|0\rangle = \mathcal{N} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} R^n |p\rangle |n-p\rangle
\]

We use the von Neumann entropy as a measure for the degree of entanglement, defined for a bipartite pure state \(\rho_{AB}\) as

\[
S_v = -\text{Tr}(\rho_A \ln \rho_A) = -\sum_{i=0}^{\infty} \lambda_i \ln \lambda_i,
\]

where \(\rho_A = \text{Tr}_B(\rho_{AB})\) is the reduced density operator of system A, and \(\lambda_i\) are its eigenvalues. Here we are interested in the degree of purity of the output mode A whose state \(\rho_A\) is obtained from \(\rho = |\text{out}\rangle\langle\text{out}|\) by tracing over the other output, mode \(B\). From Eq. (10) we obtain

\[
\rho_A = \mathcal{N}^2 \sum_{p,p'=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(m+p)! (m+p')!} |\alpha|^{2m} |\alpha|^p R^m R^{p'} |p\rangle |p'\rangle
\]

From the above equation, it is clear that the von Neumann entropy depends on the outcome of the measurement of the observable \(\alpha\). Furthermore \(S_v\), depending on the specific form of \(f\).
Then, let us consider two types of deformations in more detail. The standard deformation defined by [33, 34]

$$f_M(n) = \frac{1}{n} \frac{q^n - q^{-n}}{q - q^{-1}}, \quad q \in \mathbb{R}$$  \hspace{1cm} (13)

and the deformation given [35, 36]

$$f_A(n) = \frac{1}{n} \frac{q^n - 1}{q - 1}, \quad q \in \mathbb{R}.$$  \hspace{1cm} (14)

These deformations can be used for the characterization of the photon statistics of laser outputs reasonably close to threshold, single-atom resonance fluorescence, the micromaser field and absorption by two-level atoms.

To explore the influence of deformation parameter on the entanglement behavior of the output state in the present scheme, we have plotted in Figs. 1 and 2 the variation of the von Neumann entropy $S_v$ as a function of the parameter $q$ for various values of amplitude $|\alpha|$ for both kinds of deformations, respectively. The solid line presents the variation of $S_v$ for $|\alpha| = 1$, the dashed line is for $|\alpha| = 2$, the dotted line is for $|\alpha| = 3$ and dotted-dashed line is for $|\alpha| = 4$.

![Fig. 3. Werhl entropy of input field is plotted against the single parameter $q$ for various values of parameter $|\alpha|$ in the case of $f_M$-deformation. The solid line is for $|\alpha| = 1$, the dashed line is for $|\alpha| = 2$, the dotted line is for $|\alpha| = 3$ and dotted-dashed line is for $|\alpha| = 4$.](image)

In the case where $q$ gets farther from the undeformed $q \rightarrow 1$ limit (from left or right), the amount of entanglement increases and goes to a maximum as $q$ becomes significantly large. For certain values of the amplitude ($|\alpha| = 1$ and $|\alpha| = 2$), the von Neumann entropy obeys a sudden change in its behavior. This is due to the form of Eqs. (13) and (14) which show many singularities by varying $q$. From another side, the dependance on the amplitude shows that the increase of the input field photon number is accompanied with the increase of quantum entanglement for different values of the deformation parameter. Furthermore, the output state attains its maximal entanglement for high values of the amplitude. In this sense, the output entanglement is very sensitive to the effect of input field type and its different strength regimes. From these results, one may find that $\alpha$ and $q$ may enhance the amount of entanglement of the output state.

Now, we want to examine and explain our results using the Werhl entropy in order to investigate the physical properties of the deformed $f$-coherent states and compare them with Glauber coherent states in the considered cases. The Werhl entropy is a useful tool to investigate the different physical properties of quantum systems that contain all the information of the optical field, being completely equivalent to the density operator and it can be interpreted as an information measure for such joint measurement. The Werhl entropy clearly distinguishes coherent states and it can be used as a measure of the statistical properties of the optical fields. It conjectured that the minimum of entropy is obtained for coherent states, which are localized in the phase space, as allowed by the Heisenberg uncertainty. In this way, it presents a good mea-
sure of the strength of the coherent component in an optical field, i.e., it can be used to observe squeezing of the quantum fluctuations of the quadrature operators when the field is initially in a coherent state where it measures how much “coherence” a given state has. As it can be used to classify quantum fields with respect to their statistical properties, this measure has many advantages compared to other quantities such as Wigner function, Glauber–Sudarshan $P$ representation, etc., which take on negative values. Here, we trace the $f$-deformed coherent state from what we think as the most classical, $q = 1$, to the least classical $q \neq 1$. The Werhl entropy is defined as [37–39]

$$S_W = - \int \rho \ln \rho \, d^2 \beta,$$

where

$$Q_\beta = \frac{1}{\pi} \langle \beta | \rho | \beta \rangle,$$

and $d^2 \beta = |\beta| d|\beta| d\Theta$ is the coherent-state representation of density matrix.

In Figs. 3 and 4, we show the Werhl entropy of the $f$-deformed coherent states as a function of the parameter $q$ for different values of the $|\alpha|$ for both deformations. The Werhl entropy satisfies the inequality $S_W \geq 1 + \ln \pi$, approaches $1 + \ln \pi$ as $q$ is close to one, indicating that the deformed coherent states approach the Glauber states and reach $1 + \ln \pi$ for $q = 1$. We notice that the plots of the Werhl entropy have similar behavior as the von Neumann entropy one for the different kinds of deformations. Indeed, the $S_W$ parameter increases from the minimal value attained at $S_W = 1 + \ln \pi$ corresponding to the Glauber states (for such a value the deformed coherent states tend to factorize after beam splitting) as the deformation parameter moves away from one. This means that the $f$-deformed coherent states become more quantum mechanical than ordinary coherent states accompanied with an increase of the output state entanglement. Furthermore, the Werhl entropy increases with increasing amplitude and stabilizes at the maximal values for high values of $|\alpha|$. From the above results, the Werhl entropy of the input field is a monotonic function of the linear entropy of the output beam without a direct relation between them. Thus, for our propose, one may expect the behavior of the output state entanglement from the variation of the Werhl entropy.

**CONCLUSIONS**

We have presented a useful scheme for generating high amount of bipartite entanglement using the optical device called beam splitter. In fact, we have investigated in detail the entanglement generated via a 50:50 beam splitter when a deformed bosonic coherent state is injected on one input beam and the vacuum is injected in the other one for different strength regimes of the input optical field. In this sense, it’s shown that the $g$-deformation parameter and the photon number of the input optical field may enhance the amount of the entanglement generated at the output state. We have derived the Werhl entropy of the input field and presented the correlation between this physical quantity and the entanglement of the output beam state. In this sense, we found that the Werhl entropy may be used as an indicator of the entanglement behavior in this scheme.

Finally, the $f$-deformed coherent states provide a much richer structure than undeformed ones. They
are useful to generate and measure the entanglement and their use is not only of theoretical purpose but also of some practical importance due to their experimental accessibility. In the future, it will be important to study the entanglement generated via a beam splitter when the input state is defined in a mixed state, which make a useful contribution for more understanding the correlations process related to the beam splitter device. Another interesting line is to study the entanglement of atom-deformed field system in the cavity decay.

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