Two-photon correlation in a cascade amplifier: Propagation effects via a simple model, nonclassical regimes, and validity of neglecting Langevin noise

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We study several aspects of two-photon correlation of an optically driven extended medium (amplifier) with parametric down-conversion (PDC) scheme and three-level cascade scheme. The correlation for the PDC scheme is modeled by coupled parametric equations with constant self-coupling and cross coupling coefficients. This provides simple physical insights to how the correlation profile depends on the sign and magnitude of the coefficients in the presence of propagation group delay between signal and idler photons. The results with constant coefficients are related to the results of the driven three-level cascade scheme obtained from quantum Langevin-Maxwell’s equations. The correlation transforms from bunching to antibunching as the pump field increases. Cauchy-Schwartz inequality is violated for all time delay in the case of off-resonant pump, showing nonclassical correlation. We verify that the correlation obtained without noise operators is qualitatively correct regardless of the optical density, especially for large detuning. We also discuss the interesting physics behind antibunching and oscillations found in the reverse correlation.

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I. INTRODUCTION

The generation of a two-photon state is mainly based on a spontaneous parametric down-conversion (SPDC) process [1], where the spontaneous emission of an idler photon is followed immediately by the spontaneous emission of a signal photon. In extended medium, we have an amplifier where the parametric process has been used to generate squeezed photons [2] and entangled photon pairs in an amplifier [3]. Usually, a \( \chi^{(2)} \) crystal (such as in BBO) is used as a nondegenerate optical parametric amplifier (NOPA), where the parametric down-conversion (PDC) process produces the signal field that has a different group velocity from the idler due to a strong dispersion.

The three-level cascade scheme is closely related to the PDC scheme. When an intermediate level is a real level we have a three-level cascade scheme where the signal photon is generated within the inverse lifetime of the level. If the lifetime is longer than the lifetime of the top level \( a \), the signal photons are only weakly correlated to the idler photons. In the extended medium, the duration of quantum correlation (correlation time) in the cascade scheme is governed not only by the lifetime of the intermediate state but also by the group delay. Two-photon correlation for a single atom with the cascade scheme has been exploited for subnatural spectral linewidth [4]. The scheme has also been considered for producing entangled photon pairs through a cavity [5] and recently in quantum dots [6], but not in an extended medium (amplifier) with significant spatial propagation effect.

Here, we study the quantum correlation of the cascade scheme in extended medium using the full quantum Langevin theory [7]. We focus on the effects of propagation. In order to gain insight into the physics of the correlation, we use simple models, i.e., a parametric amplifier with constant self- and cross coefficients to help us understand the effects of relative group delay on the two-photon correlation. We first present the correlation for the single atom case in Sec. II. In Sec. III, we employ simple models for the optical parametric amplifier with constant coefficients when the idler and signal fields have different propagation velocities. We discuss the propagation effects on the correlation profile. In Sec. IV, we present the theory and the results of the three-level cascade scheme obtained by using the quantum Langevin theory, and then relate the results to the simple models, particularly for cases of resonant and off-resonant pump field. The effects of propagation, the role of noise operators, and quantum interference are discussed further in Sec. V.

II. CORRELATION FOR SINGLE ATOM

To set the stage for a more elaborate theory, we remind ourselves of the physics of correlation for single three-level cascade atom (top, middle, and bottom levels designated as \( a, b, \) and \( c \), respectively) located at \( r \) [4]. The two-photon state is [8]

\[
|\Psi\rangle = \sum_{k,q} g_{k,q} e^{-(i|k+q|)r} \left( \begin{array}{c} \nu_q + i \frac{1}{2} \Gamma \left( \nu_q - \omega_{bc} + i \frac{1}{2} \gamma \right) \end{array} \right) \left( \begin{array}{c} 1_{k,1,q} \end{array} \right),
\]

where \( \Gamma = 2 \pi |g(\omega_{ab})|^2 D(\omega_{ab}) \) is the spontaneous decay rate for (idler) photon with wave vector \( k \) and frequency \( \nu_k \) from level \( a \) to level \( b \) with the density of states \( D \) evaluated at \( \omega_{ab} \); \( \gamma = 2 \pi |g(\omega_{bc})|^2 D(\omega_{bc}) \) is the decay rate for (signal) photon with wave vector \( q \) and frequency \( \nu_q \) from level \( b \) to
level $c$. The radiation-atom couplings are $g_k = \tilde{v}_{hz} \cdot \tilde{E}_k \sqrt{\frac{2\hbar}{x_0A}}$ and $g_q = \tilde{v}_{zh} \cdot \tilde{E}_q \sqrt{\frac{2\hbar}{x_0A}}$. The first denominator in Eq. (1) corresponds to the emission of the $k$ photon that is correlated energetically to the $q$ photon. The second denominator is the emission of a second photon, i.e., the $q$ photon.

The corresponding Glauber’s two-photon correlation describing the joint detection rate by detector 1 at $r_1$, $t_1$ and detector 2 at $r_2$, $t_2$ is

$$G^{(2)}(r_1,t_1;r_2,t_2) = G^{(2)}(1,2) = |\psi(1,2) + \psi(2,1)|^2,$$  
(2)

where the partial two-photon amplitude $\psi(1,2)$ evaluated using Eq. (1) is

$$\psi(1,2) = \eta e^{-i\omega_{00} t_1} \Theta(t_1) e^{-i\omega_{00} (t_2-t_1)} \Theta(t_2-t_1),$$  
(3)

where $t_r = t_i - r_i/c$ is the emission or retarded time of the photon that goes to detector $i = 1, 2$ and $\eta$ is an uninteresting overall constant. The Heaviside step function $\Theta(t_1)$ ensures a positive emission time. The $\Theta(t_2-t_1)$ indicates that the signal photon is emitted only after the idler, where detector 1 detects the idler photon and detector 2 detects the signal photon.

The correlation shows a classical character through the exponential decay function,

$$G^{(2)}(1,2) \propto e^{-\gamma (t_r-t_i)}.$$  
(4)

We proceed to find out what would be the correlation in an extended medium.

III. CORRELATION FOR EXTENDED MEDIUM: SIMPLE MODEL

The effects of group delay in propagation can be most easily understood by considering models with constant gain or loss and cross coupling coefficients.

A. Medium without loss or gain

This section serves to provide an intuitive physics of the effect of group velocity delay on the correlation. We consider a simple model of a hypothetical medium without gain or loss (zero self-coupling coefficient) such as a crystal, which is described by the coupled parametric amplifier equations [9]

$$\left( \frac{1}{c_1} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) \hat{E}_1(z,t) = -iKE\hat{E}_1^*,$$  
(5)

$$\left( \frac{1}{c_2} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) \hat{E}_2(z,t) = iKE\hat{E}_1,$$  
(6)

where $c_1$ and $c_2$ are the velocities of the idler and signal and $K$ is the cross coupling coefficient. The assumptions for the same $K$ in both equations is justified for describing the parametric amplifier where the pump laser is far detuned from any internal levels, as supported by the analysis in Sec. IV B for the cascade scheme with a large detuning.

FIG. 1. (Color online) Correlation $G^{(2)}(\tau)$ (normalized) between the idler and signal field of parametric amplifiers without gain or loss [in Eqs. (5) and (6)] and (a) with relative group delay $\nu_i = c/10^4$ [from Eq. (11)] and (b) without relative group delay [from Eq. (12), and $\delta(\tau)$ is approximated by $\sin(\nu_{max}\tau)/\tau$] (c) Three-level cascade scheme driven by a pump field for an amplifier of length $L$. The dressed state picture for strong resonant pump depicts quantum interference of the idler field due to ac Stark splitting in levels $a$ and $c$, but no interference from the splitting of level $e$. The $n$ denotes the photon number of the pump field.

1. With relative group delay

Consider the scheme (Fig. 1) where the idler photons (subscript “1”) propagate essentially at the speed of light $c_1 = c$, but the signal photons (subscript “2”) are slowed to a constant velocity $c_2 = \nu_i \ll c$ due to the pump field.

The solutions of the Fourier transforms of Eqs. (5) and (6) via

$$\frac{\tilde{E}_1(z,\nu)}{\tilde{E}_2(z,\nu)} = \int_{-\infty}^{\infty} \left( \frac{\tilde{E}_1(z,\nu)}{\tilde{E}_2(z,\nu)} \right) e^{i\nu t} dt$$

are expressed in terms of the boundary operators,

$$\tilde{E}_1(z,\nu) = e^{i\nu t}\left[C'(\nu,z)\tilde{E}_1(0,\nu) + S'(\nu,z)\tilde{E}_1^*(0,\nu)\right],$$  
(7)

$$\tilde{E}_2(z,\nu) = e^{i\nu t}\left[C(\nu,z)\tilde{E}_1^*(0,\nu) + S(\nu,z)\tilde{E}_1(0,\nu)\right],$$  
(8)

with
\[ S(\nu, z) = iK \frac{\sin \Phi_z}{\Phi}, \] (9)

\[ C(\nu, z) = ia_+(\nu) \frac{\sin \Phi_z}{\Phi} + \cos \Phi_z, \] (10)

where \( a_+(\nu) = \frac{1}{2} (\frac{1}{\nu_1^c} + \frac{1}{\nu_2^c}) \) and \( \Phi(\nu) = \sqrt{a_-(\nu)^2 - K^2}. \)

For input vacuum fields, we may use the usual commutation relation [7] \( \hat{E}_I(z, \nu) \hat{E}_S(z, \nu) = C_\nu \delta(\nu - \nu') \), \( C_\nu = \frac{\nu g}{\nu_1^c} \), \( j = 1, 2 \), and obtain the cross correlation

\[ G_{21}^{(2)}(\tau) = \langle \hat{E}_S(z, t) \hat{E}_I(z, t + \tau) \hat{E}_S(z, t + \tau) \hat{E}_I(z, t) \rangle = N_c \left[ \int S(\nu)C^*(\nu)e^{i\nu \tau}d\nu \right]^2 + \left[ \int S(\nu)^2 d\nu \right]^2. \] (11)

where \( N_c = (Kc \frac{\nu_g}{2\pi})^2 \) is an uninteresting constant.

The plot of Eq. (11) via numerical integration is shown in Fig. 1(a), which agrees with the plot of Balic et al. [13]. We have taken a sufficiently large limit of \( \nu \) for the numerical Fourier frequency integration to minimize the oscillations near the discontinuous edges or the Gibbs phenomena [14].

This model neglects gain or loss of the fields throughout the entire propagation. Here, the velocity of the idler and signal fields propagate with constant speeds of \( c \) and \( u_\nu \) at any point in the medium. Since \( c > u_\nu \), it is safe to assume that the idler photons from any point in the medium arrive at almost the same. Thus, the correlation at time delay around \( \tau = L/u_\nu \) is due to the correlation between the idler generated from the input boundary \((z = 0)\) to the output medium \((z = L)\), and the signal generated at the input \((z = 0)\) of the medium. This shows that two photons emitted by two independent atoms as far apart as \( L \) could be correlated. By the same reasoning, the correlation at \( \tau = 0 \) is due to the idler and signal generated at the output of the medium.

2. Without relative group delay

In the absence of dispersion or pump field, the simple model [Fig. 1(b)] depicts a two-photon process where both the idler and signal photons propagate at the speed of light, \( c_{1,2} = c \) in Eqs. (7) and (8). We may also set \( \Phi(\nu) = iK \), \( a_-(\nu) = 0 \), and \( a_+(\nu) = \nu/c \) in Eqs. (9) and (10), and use Eq. (11) to obtain

\[ G_{21}^{(2)}(\tau) = \left( \frac{KC_\nu}{L} \delta(\tau) \sinh Kz \cosh Kz \right)^2. \] (12)

The correlation equation (12) is plotted numerically in Fig. 1(b), which shows a sharp peak around \( \tau = 0 \), in agreement with the \( \delta \) function of Eq. (12).

B. Medium with constant gain and loss

In the presence of constant self-coupling parameters \( G_{1,2} \), the parametric equations become

\[ \left( \frac{\partial}{\partial z} - i \frac{\nu}{\epsilon_1} + G_1 \right) \hat{E}_1 + K_1 \hat{E}_2^* = 0, \] (13)

\[ \left( \frac{\partial}{\partial z} - i \frac{\nu}{\epsilon_2} + G_2 \right) \hat{E}_1 + K_2 \hat{E}_2^* = 0. \] (14)

The correlation \( c_{12}^{(2)}(\tau) \) for Eqs. (13) and (14) with \( c_1 = c \) and \( c_2 = u_\nu \ll c \) are plotted in Fig. 2. Note that \( K_1 = -K_2 \) and \( K_1 = K_2 \) correspond to the parametric amplification and parametric oscillation regimes, respectively. The former is from the usual PDC process [9], which also corresponds to Eqs. (5) and (6), while the latter is realized for the counter-propagating idler and signal fields [15,16]. In the absence of gain and loss, the correlations are shown by the plots (i)–(viii) in Fig. 2(a). For \( K_1 = K_2 \), the “bulging” increases with the magnitude of the cross coupling coefficients [plots (i)–(iv)]. For sufficiently large value, the antibunching profile becomes visible and can be associated with parametric oscillations. Then, antibunching here may not be regarded as non-classical. In contrary, the case \( K_1 = -K_2 \) shows “denting” and bunching in the correlation profile [plots (v)–(viii) in Fig. 2(a)]. As we show in Sec. IV B, this case corresponds to the off-resonant pump field in the cascade scheme, i.e., with the microscopic complex coefficients satisfying \( K_1 = -K_2 \).

If the idler is amplified and the signal is lossy, i.e., \( G_1 < 0, G_2 > 0 \) [plots (i) and (ii) in Fig. 2(b)] the correlations profiles skew upward (maximum on right side) with time delay. The reverse occurs when the idler is lossy while the signal is amplified, i.e., \( G_1 > 0, G_2 < 0 \) [plots (iii) and (iv) in Fig. 2(b)] give the profiles that skew downward. The parameters that give skewing upward profiles may yield a larger correlation time.

C. Physical interpretations

Figure 2(c) illustrates the essential physics of how the propagation group delay affect the correlation profile. The \( G_{21}^{(2)} \) at \( \tau = L/c = 0 \), i.e., \( G_{21}^{(2)} \) is mainly due to the correlation between the idler emitted from A (or any point in the medium if \( c_2 \ll c_1 \)) and the signal emitted from C. Similarly, the \( G_{21}^{(2)} \) around \( \tau = L(1/2u_\nu - 1/c) \) \( (G_{21}^{(2)} \) ) is due to the correlation between the idler from A and the signal from B; and the \( G_{21}^{(2)} \) around \( \tau = L(1/2u_\nu - 1/c) \) \( (G_{21}^{(2)} \) ) is due to the correlation between the idler from A and the signal from A.

Since we are analyzing the correlation of the signal to the idler, the properties of the signal largely determine the correlation. The magnitude of the correlation increases with the signal. So, the skewing up feature in Fig. 2(b), plots (i) and (ii) is due to the fact that the lossy signal reduces the correlation from \( G_{21}^{(2)} \) to \( G_{11}^{(2)} \). Similar arguments apply to the skewing down feature. Also, the correlations with amplified idler in Fig. 2(b) plots (i) and (ii) are smaller than those with lossy idler in Fig. 2(b), plots (iii) and (iv). This is because a built up idler tends to accommodate a smaller correlation of the signal which follows it. As a general rule, large correlation is obtained if the preceding field (idler) is weak while the following field (signal) is strong. This observation agrees with the existing results for the spontaneous Raman-electromagnetic induced transparency scheme [7].

IV. CORRELATION FOR EXTENDED MEDIUM: QUANTUM LANGEVIN FOR THREE-LEVEL CASCADE

Although the PDC scheme provides the simplest way for generating photon pairs, the three-level cascade scheme
provides a closely related physical process with accessible microscopic theory via quantum optical methods. We employ quantum Langevin theory with one-dimensional propagation to obtain more realistic results for the correlation. The quantum dynamics of the idler field $E_{1} = A_{1}/g_{1}$ and signal field $E_{2} = A_{2}/g_{2}$ ($g_{1} = \nu_{ab}/\hbar$, $g_{2} = \nu_{bc}/\hbar$ as the coupling constants) are described by the coupled equations in frequency domain [17],

$$
\left( \frac{\partial}{\partial z} + G_{1} \right) \hat{A}_{1}(z, \nu) + K_{1} \hat{A}_{1}^{*}(z, \nu) = \tilde{F}_{1}.
$$

FIG. 2. (Color online) (a) Correlation $G_{21}(\tau)$ (arbitrary unit) versus dimensionless time delay $\tau \nu/L$ for the simple model with zero gain $G_{1} = G_{2} = 0$. As the cross coupling coefficients increase, the profile deforms with bulging for $K_{1} = -K_{2}$ and denting for $K_{1} = K_{2}$.Crudely speaking, both situations correspond to bunching and antibunching profiles, respectively. (b) Correlation using the simple model with finite gain and loss. As the cross coupling coefficients increase, the profile deforms with bulging for $K_{1} = -K_{2}$ and denting for $K_{1} = K_{2}$.(c) The physics of the above plots can be explained by the illustration. The slow group velocity of signal $\nu_{g} \approx c/10^{3}$. In the absence of relative group delay $c_{1} = c$, the gain or loss has no effect on the correlation which remains as a $\delta$ function. The $\alpha = 2g_{1}K_{1}/\gamma_{ab}$ is the absorption coefficient and $L = 1$ mm.
FIG. 3. (Color online) Two-photon self-, cross, and Cauchy-Schwartz correlations \( g_{2}^{(2)}(L, \tau) = g_{2}^{(2)}(L, \tau)/I_{2}(L) \), respectively for the three-level cascade scheme obtained with full quantum Langevin theory for resonant \( \Delta = 0 \) case. We have used \( \Omega_p = 5 \gamma_a, \gamma_p = 10^8 \text{ s}^{-1}, \gamma_p = 10^7 \text{ s}^{-1}, \text{ and } N_{0}L = 2|g_{21}(L)|L/\gamma_a \) with the cross section \( \sigma = c\omega_{ab}/\hbar c \gamma_a \gamma_b \). The correlation oscillates between nonclassical and classical regimes with time delay. It is interesting to see the correlation profile is preserved for large optical density.

\[
\frac{\partial}{\partial \tau}\hat{A}_{2}(z, \nu) + \kappa_{2}\hat{A}_{1}(z, \nu) = \overline{F}_{2}, \tag{16}
\]

with the frequency-dependent microscopic gain and loss and cross coupling coefficients, and the noise operators defined as

\[
G_{1}(\nu) = \alpha_{1}\left(-\omega^{\nu}_{aabb} - \frac{\Omega_p}{T_{bc}(\nu)}\rho_{aa} - \frac{i\nu}{c}\right), \tag{17}
\]

\[
\kappa_{1}(\nu) = \alpha_{1}\left(-\rho^{\nu}_{ac} + \frac{\Omega_p}{T_{bc}(\nu)}\rho_{abbc}^{*}\right), \tag{18}
\]

\[
\tilde{F}_{1}(z, \nu) = \alpha_{1}F\left(i\epsilon^{\nu}_{aabb}\hat{F}_{ab}^{\dagger} - \frac{\Omega_p}{T_{bc}(\nu)}e^{-i\nu t}\hat{F}_{bc}\right), \tag{19}
\]

\[
G_{2}(\nu) = \alpha_{2}\left(-\omega^{\nu}_{bbcc} + \frac{\Omega_p^{*}}{T_{ab}(\nu)}\rho_{aa}^{\nu} - \frac{i\nu}{c}\right), \tag{20}
\]

\[
\kappa_{2}(\nu) = \alpha_{2}\left(i\epsilon^{\nu}_{bbcc}\hat{F}_{ab}^{\dagger} + \frac{\Omega_p^{*}}{T_{ab}(\nu)}\rho_{aa}^{\nu}\right), \tag{21}
\]

\[
\overline{F}_{2}(z, \nu) = \alpha_{2}F\left(\frac{\Omega_p^{*}}{T_{ab}(\nu)}e^{i\nu t}\hat{F}_{ab}^{\dagger} - ie^{-i\nu t}\hat{F}_{bc}\right), \tag{22}
\]

\[
\alpha_{2} = \frac{g_{2}^{2}(\nu)}{I_{1}(\nu) + I_{p}T_{bc}(\nu)}, \tag{24}
\]

where \( F \) represents the Fourier transform of time, \( \kappa_{1} = N\rho_{ab}^{*}c\mu_{a}\nu_{c}/2 \), and \( \kappa_{2} = N\rho_{bc}^{*}c\mu_{a}\nu_{c}/2 \) are propagation constants, \( \nu_{1} \) is idler frequency, \( \nu_{2} \) is signal frequency, \( T_{ab}(\nu) = T_{ab} - i\nu \), \( T_{ab} = i(\Delta_{ab} + \gamma_a) \), \( T_{p} = |\Omega_p|^{2}, \) and \( \omega^{\nu}_{aabb} = \rho_{ac} + \rho_{bc}^{*} \) with \( \rho_{ac}^{*} \) and \( \rho_{bc} \) as the steady-state solutions of density matrix elements in zero order of the signal and idler fields, as given by

\[
\rho_{ac}^{\nu} = \frac{i\Omega_p}{T_{ac}}(2\rho_{aa}^{\nu} + \rho_{bb}^{\nu} - 1), \tag{25}
\]

\[
\rho_{aa}^{\nu} = \frac{I_{p}^{\nu}/\gamma_{1}}{1 + I_{p}^{\nu}/\gamma_{2}}, \tag{26}
\]

\[
\rho_{bb}^{\nu} = \frac{\gamma_{a} - \gamma_{p}^{*}}{\gamma_{b}}\rho_{aa}^{\nu}, \tag{27}
\]

where \( I_{ac} = \frac{1}{\gamma_{a}} + \frac{1}{\gamma_{c}} \) and \( \gamma_{p} \) is the decay rate from level \( a \) to level \( c \).

We proceed to solve the coupled equations (15) and (16) for the idler and signal fields and compute the following normalized correlations using decorrelation method [18] for paired operators and Einstein’s relation [19]. The relevant details of calculations can be found in Ref. [7], particularly Appendixes E and F therein. The self-correlation \( g_{2}^{(2)}(L, \tau) \)

\[
\Delta = 0
\]

Small optical density \( N_{0}L=0.1 \)

Large optical density \( N_{0}L=10 \)
In order to make connections with the model with constant gain and loss, we note the correspondences \( G_f \leftrightarrow \text{Re}(G_f) \) and \( K_f \leftrightarrow \text{Re}(K_f) \) (\( f = 1, 2 \)). If the population is not inverted, as usual \( w < 0 \), the idler amplifies \( (G_1 < 0) \) and propagates with the speed \( c \) while the signal propagates with \( u_2 \) (slower than \( c \)). For weak pump \( \Omega_p < \gamma \), the signal damp \( G_2 > 0 \). When the pump field is above the threshold \( \gamma \) (i.e., \( \Omega_p > \gamma \)), we have amplification without inversion of both the ider and signal \( G_2 < 0 \). Also, \( |G_1| > |G_2| > |K_1| = 2|K_2| \) and \( K_1 = 2K_2 = -i\kappa \), where \( \kappa > 0 \). The correlation that is modeled with coefficients that satisfy these conditions is plotted in Fig. 5(a).

### B. Far-detuned pump

For large detuning \( \Delta_{wa} = \Delta_{ab} = \Delta > \Omega_p, \gamma_{ac} \) and \( \Delta_{bc} = 0 \), we find \( \frac{I_p}{\gamma_{ac} - \Delta - i\kappa} \ll 1 \). Thus, we have \( \rho_{aa} \approx \rho_{bb} = 0, \rho_{cc} = 1 \), and \( \rho_{ac}^2 = \frac{g_1 g_1}{\gamma_{ac} - \Delta - i\kappa} \), which give

\[
G_1(v) = \frac{ig_1k_1}{\Delta} \left[ \frac{I_p}{(\gamma - i\nu)^2 + I_p / \gamma} - i\frac{\nu}{\gamma} \right],
\]

\[
K_1(v) = \frac{g_1 k_1}{\Delta} \left[ \frac{I_p}{(\gamma - i\nu)^2 + I_p / \gamma} - i\frac{\nu}{\gamma} \right],
\]

\[
G_2(v) = \frac{g_2}{(\gamma - i\nu)^2 + I_p / \gamma} \left[ \frac{I_p}{\gamma} - i\frac{\nu}{u_2} \right],
\]

\[
K_2(v) = \frac{g_2 k_2}{(\gamma - i\nu)^2 + I_p / \gamma} \left[ \frac{I_p}{\gamma} - i\frac{\nu}{u_2} \right],
\]

where \( u_2 = \left( \frac{1}{c} - \frac{g_2}{\gamma^2 + I_p} \right)^{-1} \) is the slow group velocity of the signal. The growth of the signal or idler is subjected to saturation due to \( I_p \) in the denominator in Eqs. (28) and (30).
using only boundary operators. Photon antibunching, but the Cauchy-Schwartz correlation exists. For large resonant pump field, Figs. 3 and 4 show bunching with values confined between 1 and 2. Nonclassical correlation. The self-correlations $g_{ff}^{(2)}$ in Figs. 3 and 4 show bunching with values confined between 1 and 2. For large resonant pump field (Fig. 3), $g_{21}^{(2)}$ shows photon antibunching, but the Cauchy-Schwartz correlation $g_{21}^{CS}$ oscillates between classical ($<1$) and quantum ($>1$) regimes with small peaks. For off-resonant pump (Fig. 4), the cross correlation $g_{21}^{CS}$ shows photon bunching. However, the $g_{21}^{CS}$ shows that the idler and signal photons are correlated nonclassically for any time delay $\tau$. Note that the $g_{21}^{CS}$ is a reliable measure of nonclassicality and not antibunching. Since the off-resonant case closely describes the parametric amplification, the results confirm the nonclassicality of the photon pairs generated commonly from $\chi^{(2)}$ crystals via PDC. Also, the degree of nonclassical correlations for both resonant and nonresonant cases are much smaller than the double $\Lambda$ scheme [7].

(b) Dependency on optical density. It is interesting to see that as the optical density $\text{NoL}$ increases the temporal extent of the correlation does not change much for the resonant case (Fig. 3), but reduces noticeably for the large detuning case (Fig. 4). It is different for the off-resonant case. Comparison of Figs. 7(a) and 7(b) reveals that propagation reduces the correlation time between the photon pairs, especially at small $\Omega_p$. Thus, the correlation time for resonant case (Fig. 6) is hardly affected by the optical density. This observation shows that a resonant pump field can maintain the correlation time, which could be useful to counteract the effect of decoherence which tends to reduce the correlation time in long propagation distance.

(c) Role of quantum noise operators. Here, we compare the correlations obtained with only boundary operators $g_{21}^{(2)b}$ with the results using the simple model and the cascade scheme with quantum theory, we can elaborate on several aspects of the underlying physics.

V. ANALYSIS OF RESULTS

Figures 3, 4, 6, and 7 show the normalized correlations for the cascade scheme computed from the solutions of Eqs. (15) and (16) using the decorrelation method and Einstein’s relation along the lines of Ref. [7]. Based on the results obtained from the simple model and the cascade scheme with quantum theory, we can elaborate on several aspects of the underlying physics.

(a) Nonclassical correlation. The self-correlations $g_{ff}^{(2)}$ in Figs. 3 and 4 show bunching with values confined between 1 and 2. For large resonant pump field (Fig. 3), $g_{21}^{(2)}$ shows photon antibunching, but the Cauchy-Schwartz correlation $g_{21}^{CS}$ oscillates between classical ($<1$) and quantum ($>1$) regimes with small peaks. For off-resonant pump (Fig. 4), the cross correlation $g_{21}^{CS}$ shows photon bunching. However, the $g_{21}^{CS}$ shows that the idler and signal photons are correlated nonclassically for any time delay $\tau$. Note that the $g_{21}^{CS}$ is a reliable measure of nonclassicality and not antibunching. Since the off-resonant case closely describes the parametric amplification, the results confirm the nonclassicality of the photon pairs generated commonly from $\chi^{(2)}$ crystals via PDC. Also, the degree of nonclassical correlations for both resonant and nonresonant cases are much smaller than the double $\Lambda$ scheme [7].

(b) Dependency on optical density. It is interesting to see that as the optical density $\text{NoL}$ increases the temporal extent of the correlation does not change much for the resonant case (Fig. 3), but reduces noticeably for the large detuning case (Fig. 4). It is different for the off-resonant case. Comparison of Figs. 7(a) and 7(b) reveals that propagation reduces the correlation time between the photon pairs, especially at small $\Omega_p$. Thus, the correlation time for resonant case (Fig. 6) is hardly affected by the optical density. This observation shows that a resonant pump field can maintain the correlation time, which could be useful to counteract the effect of decoherence which tends to reduce the correlation time in long propagation distance.

(c) Role of quantum noise operators. Here, we compare the correlations obtained with only boundary operators $g_{21}^{(2)b}$ to the results with noise operators for both a) small and b) large optical densities.
with the correct result $g_{21}^{(2)}$ obtained with both boundary and noise operators. For resonant case (Fig. 6), there is a general agreement in the profiles, i.e., both show antibunching and oscillations. But the agreement is not so good for the weak pump field. Good agreement is found for the large pump regime regardless of the optical density.

For the off-resonant case (Fig. 7), the results for $g_{21}^{(2)}$ agree very well with $g_{21}^{(1)}$ for both small and large optical densities. This justifies the neglect of noise operators in the theory concerning PDC process in parametric amplifier. These results are in contrast to the case of double $\Lambda$ scheme where decoherence between two closely spaced ground levels is the main cause of disagreement between the correlation with boundary operators and that with noise operators.

(d) **Dependency on pump and quantum interference.** The main features for resonant pump field are antibunching and oscillations (Fig. 3). The correlation transforms from bunching with exponential decay to antibunching with oscillations as the resonant pump field increases (Fig. 6). The minima of the oscillations fall at $\tau = \pi/\Omega_p$ (i.e., at $\pi/5$ in Fig. 6 for $\Omega_p = 5\gamma_{ac}$), so the frequency of the oscillations is $\tau^{-1} = \Omega_p/\pi$. The limit of the zero pump field corresponds to the single atom case where the correlation decays almost exponentially [see Eq. (11)]. When the pump is far detuned from level $\alpha$ the Rabi oscillations disappear and the correlation $g_{21}^{(2)}$ becomes bunching (Figs. 4 and 7). However, the correlation can be very large for the small pump field, especially for delay $\tau < \gamma_{ac}^{-1}$.

The antibunching and oscillation features are signatures of quantum interference, due to the beating of Autler-Townes doublet frequencies [see Fig. 1(e)] and cannot be described by the simple model (Fig. 2). The generation of idler photons involves the interference of two decay channels while no such interference is involved in the emission of the signal photons. The interference in the idler photons shows up in the correlation with the signal photons.

(e) **Connections with the simple model.** Here, we attempt to simulate certain features of the correlation using the simple model, as shown in Fig. 5. We use reasonable values for the constant coefficients based on the analysis of the actual coefficients in the resonant case (in Sec. IV A) and off-resonant case (in Sec. IV B). The skewing up feature in the resonant case and skewing down in off-resonant case explain the larger coherence time in the resonant case compared with the off-resonant case. The model is classical and therefore cannot simulate the antibunching and oscillations.

(f) **Reverse correlation.** So far, we have been analyzing the cross correlation. The probability of detecting the idler photon after detecting the signal photon is proportional to the reverse correlation $G_{12}^{(2)}(L, \tau) = \langle \hat{E}_{1}(\tau)\hat{E}_{2}(t+\tau)\rangle$, plotted in Fig. 8 in the normalized form. For resonant pump, we still see antibunching with oscillations although only the idler photons (and not the signal photons) experience quantum interference [see Fig. 1(c)]. When the pump is detuned, it is interesting to find antibunching and oscillations even though there is no quantum interference. The period of the

![Detuned $\Delta= -20\gamma_{ac}$](image)

**FIG. 7.** (Color online) Same as Fig. 6 except that pump field is off-resonant $\Delta= -20\gamma_{ac}$. The correlation is larger for weaker pump field and always shows bunching. The results without noise operators are in agreement with those with noise operators for both a) small and b) large optical densities. This justifies the neglect of noise operators in the theory concerning the generation of photon pairs in the parametric amplifier [3].

![Reverse Correlation](image)

**FIG. 8.** (Color online) Reverse correlations $G_{12}^{(2)}(L, \tau) = G_{12}^{(1)}(L, \tau)/I_{2}(L)I_{1}(L)$ versus Rabi frequency $\Omega_p$ and time delay $\tau$ for a) resonant pump and b) off-resonant pump with optical density $N\sigma L = 1$.
oscillations increases with detuning and is independent of the pump field $\Omega_p$. Physically, after the signal photon is emitted, it takes a time $\Delta^{-1}$ to drive the atom in level $c$ to level $b$ via a virtual upper level and emit an idler photon, a quasisponge and signal photons affects the two-photon correlation of pump field oscillations increases with detuning and is independent of the extended medium with constant gain and loss. Simple physical arguments are given to explain how gain or loss of the idler and signal fields can change the correlation profile. We compare the correlations based on the simple model with that of a driven three-level cascade scheme. When the pump field or the optical density is large, the correlation changes from bunching to antibunching. It is possible to control the correlation between classical and nonclassical. Hopefully, the results presented in this paper will provide useful insights and understanding of the propagation effects and quantum interference in the extended medium, as well as verifying the negligible role of quantum Langevin noise in the cascade scheme.

VI. CONCLUSION

We have shown how relative group delay between the idler and signal photons affects the two-photon correlation of spontaneous parametric down-conversion process in an extended medium with constant gain and loss. Simple physical arguments are given to explain how gain or loss of the idler and signal fields can change the correlation profile. We compare the correlations based on the simple model with that of a driven three-level cascade scheme. When the pump field or the optical density is large, the correlation changes from bunching to antibunching. It is possible to control the correlation between classical and nonclassical. Hopefully, the results presented in this paper will provide useful insights and understanding of the propagation effects and quantum interference in the extended medium, as well as verifying the negligible role of quantum Langevin noise in the cascade scheme.


[8] M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge University Press, Cambridge, 1997), Sec. 6.4 and Sec. 21B, pp. 211 and 616.

[9] For example, see Sec. 2.5 of [10], Sec. 17.1 of [11] or Sec. 9.1 of [12].

