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Preservation of Bosonic commutation relation: Explicit evaluation of quantum Langevin operator products

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ABSTRACT

The commutation relation for the field creation and annihilation operators is shown to be preserved explicitly if the transient operators products are evaluated from the quantum Langevin equations that contain noise operators with non-vanishing expectation (mean) values. The well-known Einstein relation, which is valid for the vanishing mean value, is inapplicable here. For a two-level atom initially in arbitrary state interacting with quantum vacuum radiation, detailed steps are provided that may be useful for calculations involving transient dynamics using the full quantum formalism, particularly in evaluating products involving the noise operators, the initial (time) field operator and initial atomic operators. The cross products of these operators contribute to the preservation of the commutation relation.

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1. Introduction and motivation

Commutation relations are at the core of quantum mechanics. Atomic operators must satisfy commutation relations [1] in a self-consistent framework of Heisenberg–Langevin theory. The commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\) for the free field creation and annihilation operators can be derived by modelling the free electromagnetic (EM) energy as originating from a collection of harmonic oscillators with canonical variables, usually referred to as \(\Pi\) and \(Q\). From the perspective of second quantization, the Bosonic commutation relation is connected to the indistinguishability of identical particles. These operators are related in an analogous way to the momentum \(\hat{p}\) and position \(\hat{x}\) operators of particles with mass that satisfy the usual relation \([\hat{x}, \hat{p}] = i\hbar\).

For any unitary process the commutation relation is always conserved, since
\[
[\hat{a}(t), \hat{a}^\dagger(t)] = [U(t)\hat{a}U(t)^\dagger, U(t)\hat{a}^\dagger U(t)^\dagger] = 1.
\]

We also know that the commutation relation of a single oscillator mode \(\hat{b}\) interacting with a multimode reservoir of oscillators \(\hat{\delta}_k\) [2,3] would be conserved if the noise operator \(\hat{F}\) is included, despite the Markov and Weisskopf–Wigner approximations. The oscillator has the solution of the form
\[
\hat{b}(t) = \hat{b}(0)e^{-\gamma t/2} + \int_0^t e^{-\gamma(t-t')/2}\hat{F}(t')dt'.
\]

which contains the dissipative rate \(\gamma\) and the \(\delta\) time-correlated noise operator \(\hat{F}\) for fluctuations, consistent with the fluctuation–dissipation theorem [4]. The conservation implies that the interaction between the oscillator and the reservoir is unitary.

We often speak of quantum fluctuations of the free radiation fields. However, based on the assumption that the radiation modes are statistically independent and do not couple to each other, each mode of the fields of wavevector \(k\) as described by \(\hat{a}_k(t) = \hat{a}_k(0)e^{-ikx}\) would neither experience dissipation nor fluctuations.

We now consider what happens when the radiation field interacts with a two-level atom that is prepared in arbitrary state, and derive the dynamics for \(\hat{a}_k(t)\). Although the commutation relation for the operators of each mode of the field cannot be violated in any circumstances, we may ask: How can we show explicitly that the commutation relation is preserved? The answer to this question cannot be found in the existing literature, to our knowledge, although the relevant Langevin equations for the system have been presented in Ref. [5].

We show how the commutation relation is not violated provided that the correlation between the initial field operator \(\hat{a}_k(0)\) and the noise operator \(\hat{F}(t')\) is taken into account. The detailed mathematical steps leading to the proof provide insight into the correct way of defining the noise operators and evaluating the products of operators. This is particularly useful for evaluating Glauber’s \(G(k)\) photon correlations and the study of quantum noise in a variety of quantum systems. It also serves as a consistency check on any quantum system. In the case of spin–bath system of
Ref. [6], the field is coupled to only the population difference operator $\sigma_x$ which turns out to be time invariant. Here, the proof of the preservation of the commutation relation for the field operators can be carried out explicitly. Further analysis on spin cooling was carried out in accordance to the preservation of the commutation relation.

### 2. Quantum Langevin equations

The Hamiltonian of a two-level atom in free space and with the rotating wave approximation (r.w.a.) is

$$H = \sum_{\gamma=a,b} \hbar \omega_\gamma |\gamma>|\gamma> + \sum_k (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \hbar \omega_k)$$

$$- \hbar \sum_k (\hat{g}_k \hat{a}_k \hat{p}_{ab} + \text{adj.})$$

which where $\hat{p}_{ab} = |a><b|$ and $\hat{g}_k$ is a coupling constant. The Heisenberg equation $d\hat{O}/dt = (1/\hbar)\{\hat{O},H\}$ for atomic and field operators gives the coupled equations

$$\frac{d\hat{a}_k}{dt} = -i\omega_k \hat{a}_k + i\hat{g}_k \hat{p}_{ab}$$

and the Markov approximation which corresponds to keeping the leading term $\hat{p}_{ab}(t) = \hat{p}_{ab}(0) e^{-i\hat{H}t}$ due to the smallness of the coupling constant, giving

$$\frac{d\hat{p}_{ab}}{dt} = -\Gamma \hat{p}_{ab}(t) + \hat{F}_{ab}(t)$$

where we have used the Weiskopf–Wigner approximation, $\Gamma = \sum_k |\hat{g}_k|^2 \int_0^t e^{-i\hat{H}_{ab}(t')} dt' \approx \sum_k |\hat{g}_k|^2 2\pi \delta(\Delta_k)$. Thus, $\hat{F}_{ab}(t) = \hat{p}_{ab}(0) \hat{f}(t') + \text{adj.}$

$$\hat{f}(t) = \sum_k \hat{g}_k \hat{a}_k(0) e^{-i\omega_k t}$$

Similarly

$$\frac{d\hat{p}_{bb}}{dt} = \Gamma \hat{p}_{bb}(t) - \hat{F}_{ab}(t)$$

where

$$\hat{F}_{ab}(t) = -\hat{f}^*(t) (\hat{a}_b(t) + \hat{a}_b^*(t))$$

$$\gamma_{ab} = \sum_k |\hat{g}_k|^2 \int_0^t e^{i\Delta_k (t-t')} dt' = \frac{1}{2} (\Gamma + i\zeta)$$

The complex decoherence $\gamma_{ab} \to \frac{1}{2} \Gamma$ can be made real by normalizing the Lamb shift $\zeta$ with the transition frequency $\omega_{ab}$. Using the definitions $\hat{p}_{ab}(t) = \hat{p}_{ab}(0) e^{-i\gamma_{ab} t}$ and $\hat{a}_k(t) = \hat{a}_k(0) e^{i\omega_k t}$ the coupled equations can be rewritten as in Ref. [5].

$$\frac{d}{dt} \hat{p}_{ab} = -\Gamma \hat{p}_{ab}(t) + \hat{p}_{ab}(0) (\hat{f}^*(t) + \hat{f}(t)) \hat{p}_{ab}(t)$$

$$\frac{d}{dt} \hat{p}_{ab} = \gamma_{ab} \hat{p}_{ab}(t) + \hat{f}(t) (\hat{p}_{bb}(t) - \hat{p}_{ab}(t))$$

with $\hat{F}_{ab}(t) = e^{-i\gamma_{ab} t} \hat{F}_{ab}(t) = \hat{f}^*(t) (\hat{p}_{bb}(t) - \hat{p}_{ab}(t))$, $d/dt \hat{p}_{ab} + (d/dt) \hat{p}_{bb} = 0$ and $\hat{f}(t) = i\sum_k \hat{g}_k \hat{a}_k(0) e^{-i\Delta_k t}$.

The integrations of Eqs. (15)–(16) give

$$\hat{p}_{ab}(t) = \hat{p}_{ab}(0) e^{-\Gamma t} + \int_0^t e^{-\gamma_{ab} (t-t')} \hat{p}_{ab}(t') \hat{f}^*(t') + \hat{f}(t') \hat{p}_{ab}(t') dt'$$

$$\hat{p}_{bb}(t) = \hat{p}_{bb}(0) + \hat{p}_{ab}(0) (1-e^{-\Gamma t})$$

$$\hat{p}_{ab}(t) = 2\hat{p}_{ab}(0) e^{-\Gamma t} + 2 \int_0^t e^{-\gamma_{ab} (t-t')} \hat{p}_{ab}(t') \hat{f}^*(t') + \hat{f}(t') \hat{p}_{ab}(t') dt'$$

$$\hat{p}_{bb}(t) = \hat{p}_{bb}(0) e^{-\Gamma t} + \int_0^t e^{-\gamma_{ab} (t-t')} \hat{f}^*(t') (\hat{p}_{bb}(t') - \hat{p}_{ab}(t')) dt'$$

$$\hat{a}_k(t) = \hat{a}_k(0) e^{-i\omega_k t} + \int_0^t \hat{a}_k(0) e^{-i\Delta_k (t-t')} dt'$$

where $\hat{\gamma}_{ab}(t) = \hat{p}_{ab}(t) - \hat{p}_{ab}(0)$.

The exact solutions would involve an infinity of harmonic oscillator equations. Note that $<\hat{F}_{ab}(t)> = 0$ since $<\hat{a}_k^\dagger(0) \hat{p}_{ab}(t)> = 0$ although the initial operators are decorrelated $<\hat{a}_k^\dagger(0) \hat{p}_{ab}(0)> = <\hat{a}_k^\dagger(0) \hat{p}_{ab}(0)> = 0$.

### 3. Proving the commutation relation

Since the noise operators in Eqs. (15)–(16) have non-vanishing mean, i.e. $<\hat{F}_{ab}(t)> = 0$, we may not use Einstein’s relation, whose validity is only for zero-mean noise operators. By solving Eqs. (15) and (17) simultaneously we have the exact solutions

$$\hat{a}_k(t) = \hat{a}_k(0) e^{i\Delta_k t} - i \hat{g}_k \int_0^t \hat{a}_k(0) e^{-i\Delta_k (t-t')} dt' + \int_0^t \hat{a}_k(0) e^{-i\Delta_k (t-t')} dt'$$

where $\hat{\psi}(t) = e^{i\Delta_k t} - e^{-i\gamma_{ab} t}$ and $\hat{T}_{ab} = \gamma_{ab} + i\Delta_k$.

The required commutation is computed as follows:

$$[\hat{a}_k(t),\hat{a}_k^\dagger(0)] = [\hat{a}_k(0),\hat{a}_k^\dagger(0)] + K [\hat{\psi}(t)^2 (\hat{p}_{bb}(0) - \hat{p}_{ab}(0))$$

$$+ K \hat{F}_{ab}(t) \hat{f}(t) \hat{p}_{bb}(t), \hat{F}_{ab}(t)] dt' + \text{adj.}$$

$$-i \hat{g}_k \int_0^t e^{i\Delta_k (t-t')} \hat{a}_k(0) \hat{T}_{ab}(t') dt' + \text{adj.}$$

where $K = \int_0^t |\hat{g}_k|^2 \Delta_k^2 \Delta' e^{-i\Delta_k (t-t')} dt' + \text{adj.}$

Direct evaluation of the noise commutator using $\hat{T}_{ab} = \hat{f}^*(t) (\hat{p}_{bb}(t) - \hat{p}_{ab}(t))$ and $\hat{T}(t) = \int_0^t \hat{a}_k(0) e^{-i\Delta_k t'} dt'$ gives

$$\hat{F}_{ab}(t) \hat{T}(t) = \sum_k |\hat{g}_k|^2 (l + \hat{n}_k(0)) e^{i\Delta_k (t-t')} - \int_0^t \int_0^t \hat{a}_k^\dagger(0) e^{i\Delta_k (t-t')} dt'$$

$$\hat{T}(t') \hat{F}_{ab}(t') = \sum_k |\hat{g}_k|^2 e^{i\Delta_k (t-t')} = \delta(t-t')$$

where we use $\sum_k |\hat{g}_k|^2 e^{i\Delta_k (t-t')} = \delta(t-t')$ and $\hat{T}_{ab}(t) = \hat{T}_{ab}(0) + \hat{T}_{ab}(0) = \hat{T}$.
that Einstein’s relation also gives the same result (coincidentally) although the relation is valid for zero-mean noise operators. We have \( \{\hat{p}_a^0(0), \hat{A}_k(t)\} = [\hat{d}_k(0), \hat{\rho}_{ab}(0)] = 0 \) due to the fact that the initial operators for the atomic and the field subsystems are independent. We need to evaluate

\[
[\hat{A}_k(0), \hat{F}_{ab}(t)] = \sum_k g_k^a e^{i\omega_k t}[\hat{a}_k(0), \hat{a}^+_k(0)] \hat{p}_k(t)
\]

\[
= -i g_k^a e^{i\omega_k t}[\hat{p}_k(0) + \hat{\rho}_{ab}(0)(1 - 2 e^{-\gamma t})] = -i g_k^a e^{i\omega_k t}[1 - 2 \hat{\rho}_{ab}(0) e^{-\gamma t}]
\]  

where we use Eq. (19) to obtain the explicit form \( \hat{F}_{ab}(t) = \int t^0(\hat{p}_k(0) + \hat{\rho}_{ab}(0)(1 - 2 e^{-\gamma t} + \hat{O}) \) with \( \hat{O} \) is the integral term in Eq. (19) which does not contribute to Eq. (25) due to the odd number of radiation operators. We also evaluate

\[
[\hat{p}_a(0), \hat{F}_{ab}(t)] = 2\hat{p}_a(0) e^{-\gamma t} \sum_k g_k^a e^{i\omega_k t} = -2 e^{-\gamma t} \hat{\rho}_{ab}(0) \hat{p}_a(t)
\]

(26)

The expectation of \( \{\hat{p}_a(0), \hat{F}_{ab}(t)\} \) vanishes since \( \langle \hat{F}(t) \rangle = 0 \).

Inserting all the terms, we find that Eq. (23) becomes

\[
\hat{A}_k(t), \hat{A}_k(t)^\dagger = I - K \hat{p}_a(t)^2 \hat{\rho}_{ab}(t) + K^\prime \int_0^t |\hat{p}(t)|^2 dt + \frac{g_k^a g^a_k}{T_0} \int_0^t \hat{p}(t) e^{-\gamma(t-t')} \hat{p}_a(t') dt' + \text{adj}
\]

(27)

Using Eq. (25), the last term can be written as \(-i g_k^a e^{i\omega_k t} / T_0 \int_0^t \hat{p}(t) e^{-\gamma(t-t')} \hat{p}_a(t') dt' + \text{adj} \). Replacing \( \hat{p}(t) = e^{\gamma t} e^{-\gamma t} \hat{p}_a(t) \), the commutator \( \{\hat{d}_k(0), \hat{d}_k(t)\} \) becomes

\[
1 + K \left[ -1 + e^{-2\gamma t} - 2 e^{-2\gamma t} \cos \Delta t \right] \hat{p}_a(0) + \int_0^t \left[ 1 + e^{-2\gamma t} - 2 e^{-2\gamma t} \cos \Delta t \right] dx - \int_0^t \left( T_0 (1 - e^{-\gamma t}) - c.c. \right) \hat{p}_a(0) 2 e^{-\gamma t} dx
\]

(28)

We use \( T_0 = \gamma_{a} + i\Delta \) to expand \( T_0 (1 - e^{-\gamma t}) = T_0 (1 - e^{-\gamma t}) \). From \( \gamma_a = \gamma / 2 \) the commutator can be written as

\[
1 + K \left[ -1 + e^{-\gamma t} - 2 e^{-\gamma t} \cos \Delta t \right] \hat{p}_a(0) + \int_0^t \left[ 1 + e^{-2\gamma t} - 2 e^{-2\gamma t} \cos \Delta t \right] dx - \int_0^t \left( T_0 (1 - e^{-\gamma t}) - c.c. \right) \hat{p}_a(0) 2 e^{-\gamma t} dx
\]

(29)

where \( G_{\Delta t}(x) = e^{-2\gamma t} \cos \Delta t x + 2 \Delta e^{-\gamma t} \sin \Delta t x \).

After lengthy but straightforward algebra of evaluating the simple integrals, we find that the commutation relation is preserved, i.e.

\[
1 + K \left[ -1 + e^{-\gamma t} - 2 e^{-\gamma t} \cos \Delta t \right] \hat{p}_a(0) + \int_0^t \left[ 1 + e^{-2\gamma t} - 2 e^{-2\gamma t} \cos \Delta t \right] dx - \int_0^t \left( T_0 (1 - e^{-\gamma t}) - c.c. \right) \hat{p}_a(0) 2 e^{-\gamma t} dx = 1
\]

(30)

4. Connection with density matrix equations

Before discussing the results, we note that an alternative set of equations can be obtained by rewriting the coupled operator equations such that the defined noise operators have zero mean. The equations are then inserted back into the terms with \( \hat{F}(t) \) and use the Markov approximation again \( \hat{p}_a(0)^2 = \hat{p}_a(0) \hat{p}_a(t) \), giving

\[
\frac{d}{dt} \hat{p}_a(t) = -\Gamma (1 + \hat{\bar{n}}_a) \hat{p}_a(t) + \Gamma \hat{\bar{n}}_a \hat{p}_a(t) + \hat{F}_{ab}(t)
\]

(31)

\[
\frac{d}{dt} \hat{p}_a^0(t) = -\Gamma (1 + 2 \hat{\bar{n}}_a) \hat{p}_a^0(t) - \Gamma + 2 \hat{F}_{ab}(t)
\]

(32)

\[
\frac{d}{dt} \hat{p}_{ab}(t) = -\hat{F}_{ab}(t) \hat{p}_{ab}(t) + \hat{F}_{ab}(t)
\]

(33)

where \( \hat{n}_a = \hat{a}_a^0(0) \hat{a}_a(0) \) and \( \hat{F}_{ab}(t) = \frac{1}{2} \{ \hat{F}(t) + i \frac{\partial}{\partial t} \hat{F}(t) \} \) is the complex decoherence. The zero-mean noise operators are

\[
\hat{F}_{ab}^0(t) = \hat{\rho}_{ab}(0) \hat{F}(t) e^{-\gamma t} + \hat{F}(t) \hat{\rho}_{ab}(0) e^{-\gamma t}
\]

(34)

\[
\hat{F}_{ab}^0(t) = \hat{F}(t) \hat{\rho}_{ab}(0) + \hat{\rho}_{ab}(0)(1 - 2 e^{-\gamma t})
\]

(35)

\[
\hat{F}(t) = \sum_k g_k^a \hat{a}_k(0) e^{-\gamma t}
\]

(36)

The noise \( \hat{F}_{ab}^0 \) for the population operators vanishes for large times. The atomic operators are found to obey the fluctuation-dissipation theorem. In deriving Eqs. (31)–(33), we have used the fact the noise operators are delta-correlated, \( \int \hat{F}(t) \hat{F}(t') dt' \approx \sum_k g_k^a \hat{a}_k(0) e^{-\gamma(t-t') \gamma} \) and

\[
\int \hat{F}(t) \hat{F}(t') dt' \approx \sum_k g_k^a \hat{a}_k(0) \left\{ \pi \hat{a}(t) - i \frac{\partial}{\partial \lambda} \right\} = \frac{1}{2} \hat{n}_a (I + i \lambda)
\]

(37)

\[
\int \hat{F}(t) \hat{F}(t') dt' = \frac{1}{2} \hat{n}_a (I + i \lambda)
\]

(38)

We have tried to use Eqs. (31)–(33) to prove the commutation relation. However, the radiation operator \( \hat{n}_a \) in Eqs. (31)–(33) makes the operator equations nonlinear, and this complicates the process of proving the commutation relation. The quantum average of the above equations (31)–(33) are exactly the density matrix equations (obtained from the master equation for density operator) since \( \langle \hat{F}_{ab}^0(0) \rangle = 0 \) \( \langle \hat{\rho}_{ab}(0) \rangle = 0 \) and \( \langle \hat{\bar{n}}_a \hat{p}_{ab}(t) \rangle = \pi \hat{a}(\hat{F}(t)) \). The density matrix equations are linear and can be solved exactly.

5. Discussions

We have shown explicitly how the commutation relation for the creation and annihilation operators of the radiation field is preserved when the field interacts with a two-level atom in arbitrary state. The explicit proof is by no means trivial. All possible paired operators, especially the cross terms need to be evaluated in order to complete the proof. The calculation presented here serves as a valuable guide to evaluate the various correlations through paired operator products in more complicated multilevel systems. In Ref. [7], the cross term, i.e. the product of initial field operator and the noise operator is neglected on assumption that the initial field is decorrelated from the noise operator, leading to a violation of the commutation relation. The commutation is preserved when the cross term is included. It shows the importance of keeping the initial operators and evaluating every possible pairs in the field solution equations (22)–(21). Thus, the present work establishes a firm foundation for exact evaluation of operator products in the transient regime [8].

Please note that in the above calculations, we do not use the Einstein’s relation

\[
\frac{d(\hat{A} \hat{B})}{dt} = \hat{A} \left\{ \frac{\partial}{\partial t} - \hat{F}_1 \right\} - \hat{B} \frac{\partial}{\partial t} \hat{F}_1 = 2D
\]

which applies not only if the noise operators are delta correlated, but also if the noise operators have zero mean value. The latter does not hold in Eqs. (18)–(21) which were used to evaluate the commutator for the creation and annihilation operators. If we had used the quantum Langevin equations (31)–(33) containing noise
operators with zero mean value, we would have
\[
\langle \hat{F}_{ab}(t)\hat{F}_{ab}(t') \rangle = \delta(t - t')\Gamma(1 + \bar{n}_o)p_{ab}(t) - \Gamma\bar{n}_o\rho_{ab}(t)
\]
\[
= \delta(t - t')\Gamma(1 + \bar{n}_o)
\]
\[
(39)
\]
and hence \[\hat{F}_{ab}(t)\hat{F}_{ab}(t') = \delta(t - t')\Gamma\] (same as Eq. (24)), where we have used \[\hat{\rho}_{aa}(t) + \hat{\rho}_{bb}(t) = \hat{\rho}_{aa}(0) + \hat{\rho}_{bb}(0) = I\]. Thus, the result obtained via Einstein relation agrees with the result in Eq. (24), although the relation is derived based on the assumption that the noise operators have zero mean. The expression for \[\langle \hat{\rho}_{aa}(0)\rangle\] is the same as Eq. (25). However, the commutation relation is not conserved since \[\gamma_{ab} - \hat{F}_{ab} = \frac{1}{2}\Gamma(1 + 2\bar{n}_o)\] is not equal to \[\frac{1}{2}\Gamma\] unless \[\bar{n}_o = 0\]. It is important to note that the use of quantum Langevin equations (31)–(33) containing noise operators with zero mean value, along with the Einstein relation, leads to difficulties. The crux of the problem lies in the formidable (nonlinear) terms, \[\bar{n}_o\rho_{aa}\] and \[\bar{n}_o\rho_{bb}\].

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