Single-mode and intermodal higher-order nonclassicalities in two-mode Bose-Einstein condensates

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An analytic operator solution of a generalized quantum mechanical Hamiltonian of two-mode Bose-Einstein condensates (BECs) is obtained for short-time dynamics and the same is used to investigate the nonclassical properties of the modes present in the system. Nonclassical characters are observed by means of single-mode and intermodal quadrature squeezing, single-mode and intermodal sub-Poissonian boson statistics, and intermodal entanglement. In addition to the traditionally studied lower-order nonclassical properties, signatures of higher-order nonclassical characters of two-mode BEC systems are also obtained by investigating the possibility of higher-order antibunching and higher-order entanglement. The mutual relation among the observed nonclassicalities and their evolution (variation) with time and the ratio of the single boson tunneling amplitude (ε) and the coupling constant for the intramodal interaction (κ) are also reported.

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I. INTRODUCTION

With the rapid development of the interdisciplinary field of quantum computation and communication, a nonclassical nature of quantum state has become a mandatory requirement for application of the quantum state itself to various useful purposes. For example, nonclassical states are required to implement a family of protocols of discrete [1] and continuous variable quantum cryptography [2], quantum teleportation [3], dense coding [4], etc. Recently, several possibilities for implementation of quantum computing devices using Bose-Einstein condensate (BEC) -based systems have been reported [5–10]. For example, Josephson charged qubits are realized in two weakly coupled BECs that are confined in a double-well trap [5], a scheme for implementation of quantum algorithms using BEC is proposed [6], a Josephson qubit suitable for a scalable integrated quantum circuit is demonstrated [7], a quantum state is transferred using cavities containing two-component BECs coupled by an optical fiber [8], and schemes for implementing protocols of quantum metrology using two-component BECs are proposed [9–11]. Interestingly, many of the recently reported applications of BECs in quantum information processing involve two-mode BECs ([8–11], and references therein). Recent studies on quantum properties of BECs also include the experimental realization of multiparticle entanglement and a spin-squeezed state on a two-layer atom chip with two-component BECs [11], the generation of continuous variable entanglement in BECs through homodyne detection of atomic quadratures [12], demonstration that spin-squeezed states and entanglement in BECs improve interferometric precision [13], and the demonstration of the application of the entanglement produced using two-mode BECs in achieving sensitivity beyond shot noise in matter-wave interferometers [14]. These facts have motivated us to systematically investigate the nonclassical properties of a class of two-mode BEC systems.

A state is called nonclassical if its Glauber-Sudarshan $P$ function is negative or more singular than the δ function. However, in general, $P$ function is not directly observable through experiments. This fact led to the construction of several other criteria of nonclassicality. For example, zeros of $Q$ function, negativity of Wigner function, Fano factor, $Q$ parameter, etc., are often used to characterize nonclassicality.

In the present paper we will restrict ourselves to a group of experimentally realizable nonclassical criteria that can characterize a set of nonclassical characters of practical relevance, such as antibunching, intermodal antibunching, squeezing, intermodal entanglement, etc. Recently, a number of experimental observations of higher-order nonclassicalities are reported in quantum optical systems [15–17], and it is shown that in the case of a weak nonclassicality it may be easier to characterize the nonclassicality by means of a higher-order nonclassical criterion (Fig. 4 of Ref. [15]). Prior to these experimental studies several predictions of higher-order nonclassicality in quantum optical systems were present [18–20]. However, except for a recent work by Perinova et al. [21], no serious effort has yet been made to investigate the possibility of higher-order nonclassicalities in coupled BEC systems. Keeping this in mind we will also study the possibilities of observing higher-order antibunching and higher-order entanglement in two-mode BEC systems. A specific reason to systematically study these types of nonclassicality lies in the fact that quantum states with these nonclassical characters are already shown to be useful for various important tasks related to quantum information processing. For example, squeezed state is known to be useful for continuous variable quantum cryptography [2], teleportation of coherent states [22], teleportation of wave function of a single mode of the electromagnetic field [23], etc.; entanglement is well known as

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one of the most important resources in quantum information and it is necessary for quantum teleportation, dense coding, and many other related tasks [24]; and antibunching is known to be useful in building single-particle (single photon) sources [25]. Thus it is motivating enough to investigate the possibility of observing antibunching, intermodal antibunching, squeezing, and intermodal entanglement in two-mode BEC systems. The present paper aims to do that.

The interest in BECs has been considerably amplified in recent years with the experimental realizations of condensation. In the last two decades several groups have demonstrated BECs in ultracold dilute alkali-metal gases using magneto-optical traps ([26], and references therein). These demonstrations amplified the interest in BECs and different aspects of it have been studied in the recent past. Among these different aspects of BECs, nonclassical properties of two-mode BECs ([27–34], and references therein) have recently been studied in detail for the reasons described above. In a two-mode BEC system each mode is a BEC and bosons are restricted to occupying one of the two modes. Bosons from one mode can go to the other mode. There exist different types of two-mode BEC systems. In the present investigation we restrict ourselves to two-mode atom-atom BEC systems. Even for two-mode atom-atom BEC systems several Hamiltonians are reported in the literature [28–34]. Here we have shown that most of those Hamiltonians can be obtained from a more general Hamiltonian of two-mode BECs. Equations of motion corresponding to a two-mode BEC Hamiltonian can be solved by using different methods, such as the short-time approximation, Gross-Pitaevskii approximation, Bethe ansatz method, etc. In the present work a second-order analytic operator solution of the generalized Hamiltonian is obtained using a technique developed by some of the present authors [35–39]. The solutions obtained as time evolution of annihilation and creation operators of different modes are subsequently used to show different signatures of lower-order and higher-order nonclassicality in two-mode BECs.

The present paper is organized as follows. In Sec. II we briefly introduce the model Hamiltonian that describes the two-mode BEC of our interest. We also provide a perturbative solution of the equation of motion corresponding to this Hamiltonian. In Sec. III we show that squeezing of quadrature variables is possible for all the individual and coupled modes of the two-mode BEC. Similarly, in Sec. IV we show that antibunching can be observed for all the individual and coupled modes of the two-mode BEC system studied here. In Sec. V we study quantum entanglement using three different inseparability criteria and observe intermodal entanglement. In Sec. VI we extend the domain of the present study to the investigation of higher-order nonclassicalities and report the existence of higher-order antibunching and higher-order entanglement in two-mode BEC systems. Finally the paper is concluded in Sec. VII.

II. THE MODEL HAMILTONIAN

A general Hamiltonian for repulsive two-mode BECs can be written as

\[
H = \frac{\kappa}{4} (a^\dagger a^2 + b^\dagger b^2) - \frac{\Delta \mu}{2} (a^\dagger a - b^\dagger b) - \frac{\epsilon}{2} (a^\dagger b + b^\dagger a),
\]

where \(a(a^\dagger)\) and \(b(b^\dagger)\) are the single-particle annihilation (creation) operators for the two modes A and B, respectively, and they obey the well-known bosonic commutation relation. Throughout our present study, we consider \(\hbar = 1\). The parameter \(\epsilon\) denotes the single atom tunneling amplitude, the difference in the chemical potential between the wells (modes) is denoted by \(\Delta \mu\), and \(\kappa\) denotes the coupling constant for the atom-atom interaction. In the present study we consider only the positive values of \(\kappa\) as the interaction between atoms is repulsive in nature. Total particle number \(a^\dagger a + b^\dagger b\) is a conserved quantity and is set to a constant value \(N\). Hamiltonian (1) is general in the sense that Hamiltonians used in several existing studies [29,30,32] can be obtained from this Hamiltonian either as a limiting case or by adding constant terms that do not affect the description of any physical phenomena. Specifically, if we consider that the difference of chemical potentials between two modes is zero (i.e., \(\Delta \mu = 0\), \(\epsilon = -2\kappa\) and \(\kappa = 2g\)), then the Hamiltonian (1) reduces to the Hamiltonian used by Opanchun et al. [32]. Further, the addition of constant terms \(\frac{g}{2} (a^\dagger a + b^\dagger b)\) and \(\frac{g}{2} (a^\dagger a + b^\dagger b)^2\) would reduce the Hamiltonian (1) to the Hamiltonians studied by Hines [29] and Leggett [30], respectively.

The Hamiltonian (1) describes both the double-well BEC (external Josephson effect) and the two-level BEC (internal Josephson effect) [34]. A schematic diagram depicting a double-well BEC system is shown in Fig. 1. Such a double-well scheme seems to be experimentally realizable as has been demonstrated with magnetic potential in an atom chip to achieve fluctuations below shot noise level in BECs [40]. Two-mode BEC systems can be classified in two categories: a double-well BEC system and a single-well two-level BEC system. These two types of physical systems are similar with the only difference between them being that in the double-well BEC systems the intermodal coupling is considered very small, whereas in the single-well two-level BEC systems no such restriction is applied. To be precise, in the case of internal Josephson effect (i.e., in a single-well two-level BEC system) two BEC modes are not spatially separated, rather they are only separated by some internal (nongeometrical) quantum number, whereas in external Josephson effect two modes are spatially separated [29–31]. For example, internal Josephson effect can be observed in a system composed of Bose-Einstein condensed atoms with two different internal hyperfine levels [1] and [2].
and a laser field that can induce Raman transition $|1⟩ \rightarrow |2⟩$ ([31], and references therein). In our present study, we consider the generalized Hamiltonian (1) for two-mode Bose-Einstein condensates, and in order to study the various nonclassical features in two-mode BECs, we need the solutions of the following Heisenberg equations of motion corresponding to Hamiltonian (1):

$$
\dot{a}(t) = -i \left( \frac{\kappa}{2} a^\dagger(t) a^2(t) - \frac{\Delta \mu}{2} a(t) - \frac{\epsilon}{2} b(t) \right),
$$

$$
\dot{b}(t) = -i \left( \frac{\kappa}{2} b^\dagger(t) b^2(t) + \frac{\Delta \mu}{2} b(t) - \frac{\epsilon}{2} a(t) \right). \tag{2}
$$

The above set of coupled nonlinear differential equations of the field operators is not exactly solvable in closed analytical form. Here we use the perturbative solutions which are more general than the well known short-time approximation [19]. The technique used here is available in our previous papers [35–39].

The solutions of Eqs. (2) assume the following form:

$$
a(t) = f_1(a(0) + f_2 b(0) + f_3 a(0) a^2(0) + f_4 a(0) a^2(0) + f_5 a(0) a^2(0) + f_6 a(0) a^2(0) + f_7 a(0) a^2(0) + f_8 a(0) a^2(0) + f_9 a(0) a^2(0) + f_{10} a(0) a^2(0)),
$$

$$
b(t) = g_1 b(0) + g_2 a(0) + g_3 y a(0) b(0) + g_4 a(0) b(0) + g_5 a(0) b(0) + g_6 a(0) b(0) + g_7 a(0) b(0) + g_8 a(0) b(0) + g_9 a(0) b(0),
$$

where the parameters $f_i (i = 1, 2, \ldots, 9)$ and $g_i (i = 1, 2, \ldots, 9)$ are evaluated from the dynamics under the initial conditions.

In order to apply the initial boundary condition we put $t = 0$ in Eq. (3). It is clear that $f_i(0) = g_i(0) = 1$ and $f_i(0) = 0$ (for $i = 2, 3, 4, 5, 6, 7, 8$, and 9). Under these initial conditions the corresponding solutions for $f_i(t)$ and $g_i(t)$ are given by

$$
f_1(t) = g_1^*(t) = e^{-\gamma t},
$$

$$
f_2(t) = -g_2^*(t) = \frac{\epsilon}{2 \Delta \mu} G(t) f_1(t),
$$

$$
f_3(t) = -g_3^*(t) = -i \frac{\kappa \epsilon}{2} f_1(t),
$$

$$
f_4(t) = g_4^*(t) = \left[ \frac{i \kappa \epsilon}{4 \Delta \mu} - \frac{\epsilon^2}{4 \Delta \mu^2} G(t) \right] f_1(t),
$$

$$
f_5(t) = g_5^*(t) = -\frac{\kappa^2 \epsilon^2}{8} f_1(t),
$$

$$
f_6(t) = g_6^*(t) = -\frac{\kappa^2 \epsilon^2}{8} f_1(t),
$$

$$
f_7(t) = g_7^*(t) = \left[ -i \frac{\kappa \epsilon t}{4 \Delta \mu} - \frac{\kappa \epsilon}{4 \Delta \mu^2} G(t) \right] f_1(t),
$$

$$
f_8(t) = g_8^*(t) = \left[ -i \frac{\kappa \epsilon t}{2 \Delta \mu} + \frac{\kappa \epsilon}{2 \Delta \mu^2} G(t) \right] f_1(t),
$$

$$
f_9(t) = g_9^*(t) = \left[ \frac{i \kappa \epsilon}{4 \Delta \mu} e^{-\Delta \mu t} - \frac{\kappa \epsilon}{4 \Delta \mu^2} G(t) \right] f_1(t),
$$

where $G(t) = (1 - e^{-\Delta \mu t})$. These solutions are valid up to the second order in $\kappa$ and $\epsilon$ provided $\kappa t < 1$ and/or $\epsilon t < 1$ such that the perturbation theory is respected. These solutions are used to investigate the various nonclassical effects in the atom-atom BEC. In what follows, for our convenience we have not shown the time dependence of $f_i(t)$ and $g_i(t)$. In short, the convention used in the remaining equations is $f_i(t) = f_i$ and $g_i(t) = g_i$.

### III. QUADRATURE SQUEEZING

In order to investigate the nonclassical effects in atom-atom BEC we consider that all the atomic modes are initially coherent. Therefore, a composite coherent state arises from the product of the coherent states $|\alpha⟩$ and $|\beta⟩$ which are eigenkets of $a$ and $b$, respectively. Thus the initial composite state is

$$
|\psi(0)⟩ = |\alpha⟩ \otimes |\beta⟩. \tag{5}
$$

The field operator $a(t)$ operating on such multimode coherent state gives rise to complex eigenvalue $\alpha(t)$. Hence, we have

$$
\alpha(0)|\psi(0)⟩ = \alpha|\alpha⟩ \otimes |\beta⟩, \tag{6}
$$

where $|\alpha|^2$ is the initial number of the atoms in the state $a$. In a similar fashion $\beta(t)$ corresponds to the other atomic mode operator $b$. Now, in order to study the squeezing effects in the various modes, we define the quadrature operators

$$
X_a = \frac{1}{2} [a(t) + a^\dagger(t)], \quad Y_a = -i \frac{1}{2} [a(t) - a^\dagger(t)], \tag{7}
$$

where $a(a^\dagger)$ is the annihilation (creation) operator of the $a$ mode, which satisfies $[\alpha,a^\dagger] = 1$. Squeezing in mode $a$ is possible if the fluctuation in one of the quadrature operators goes below the minimum uncertainty level, i.e., if

$$
(\Delta X_a)^2 < \frac{1}{4} \quad \text{or} \quad (\Delta Y_a)^2 < \frac{1}{4}. \tag{8}
$$

The quadrature fluctuation in mode $a$ can be obtained by using (3)–(7) as follows:

$$
\left( \frac{\Delta X_a}{\langle Y_a \rangle^2} \right)^2 = \frac{1}{4} \left[ 1 + 2|f_3|^2|\alpha|^4 \pm \left( f_1 f_3 + f_1 f_5 |\alpha|^2 \right) \right],
$$

$$
\left( \frac{\Delta Y_a}{\langle X_a \rangle^2} \right)^2 = \frac{1}{4} \left[ 1 + 2|f_3|^2|\alpha|^4 + 3|f_3|^2|\alpha|^2 \right]. \tag{9}
$$

where c.c. stands for complex conjugate and the upper and lower signs of Eq. (9) correspond to $(\Delta X_a)^2$ and $(\Delta Y_a)^2$, respectively. In a similar manner, the quadrature fluctuation...
FIG. 2. (Color online) Plot of quadrature fluctuations versus rescaled interaction time $\kappa t$ of (a) pure mode $a$ using approximate analytic solution and exact numerical solution; (b) pure mode $b$ using approximate analytic solution and exact numerical solution; (c) coupled mode $ab$ using approximate analytic solution and exact numerical solution, for $\kappa = 10 \text{ Hz}, \Delta \mu = 50 \text{ Hz},$ and $\alpha = \beta = 5$; (d) variation of quadrature fluctuations in mode $a$ with $\kappa$ using analytic solution for $\varepsilon = 500 \text{ Hz}, t = 0.001 \text{ s}, \Delta \mu = 10^4 \text{ Hz},$ and $\alpha = \beta = 5$; (e) quadrature fluctuations of mode $a$ with time $t$ in seconds using analytic solution for $\kappa = 5 \text{ Hz}$ (smooth line), $8 \text{ Hz}$ (dashed line), and $10 \text{ Hz}$ (dot-dashed line) for $\varepsilon = 500 \text{ Hz}, \Delta \mu = 10^4 \text{ Hz},$ and $\alpha = \beta = 5$; (f) quadrature fluctuations of mode $a$ with time $t$ in seconds using analytic solution for $\varepsilon = 0 \text{ Hz}$ (smooth line), $250 \text{ Hz}$ (dashed line), and $500 \text{ Hz}$ (dot-dashed line) for $\kappa = 10 \text{ Hz}, \Delta \mu = 10^4 \text{ Hz},$ and $\alpha = \beta = 5$. In (a)–(d) the solid line represents analytically obtained $(\Delta X_i)^2$ and the dashed line denotes analytically obtained $(\Delta Y_i)^2$ with $i \in \{a, b, ab\}$. Exact numerical values of $(\Delta X_i)^2$ and $(\Delta Y_i)^2$ are marked with circles and squares, respectively.

For mode $b$ is obtained as

\[
\begin{bmatrix}
(\Delta X_b)^2 \\
(\Delta Y_b)^2
\end{bmatrix} = \frac{1}{4} \left[ 1 + 2|g_3|^2 |\beta|^4 \pm \left( g_1 g_3 + g_1 g_5 \right) |\beta|^2 \right. \\
+ g_1 g_5 |\alpha| |\beta|^2 + 3g_3^2 |\beta|^2 + \text{c.c.} \right]
\]

\tag{10}

Now we may study the intermodal squeezing in coupled mode $ab$ by using the following quadrature operator for the coupled mode introduced by Loudon and Knight in Ref. [41]:

\[
X_{ab} = \frac{1}{2\sqrt{2}} \left[ a(t) + a^\dagger(t) + b(t) + b^\dagger(t) \right],
\]

\[
Y_{ab} = -\frac{i}{2\sqrt{2}} \left[ a(t) - a^\dagger(t) + b(t) - b^\dagger(t) \right].
\tag{11}

The physical meaning of the two-mode quadrature operators defined above is the same as that of the usual single-mode quadrature operators, with the only difference being that it is applicable to two-mode bosonic systems. Using (3)–(6) and (11) we can obtain the second-order variance of coupled mode $ab$ as

\[
\begin{bmatrix}
(\Delta X_{ab})^2 \\
(\Delta Y_{ab})^2
\end{bmatrix} = \frac{1}{4} \left[ 1 + |f_3|^2 (|\alpha|^4 + |\beta|^4) \right. \\
+ \frac{1}{2} \left( (f_1 f_3 + f_1 f_5 + 2f_2 g_9) \alpha \right. \\
\left. + (g_1 g_3 + g_1 g_5 + 2g_1 f_5) \beta^2 + (f_1 f_3 + g_1 g_5) |\alpha| \beta \\
\left. + 3f_2^2 (|\alpha|^2 |\alpha|^2 + |\beta|^2 |\beta|^2 + \text{c.c.}) \right].
\tag{12}
\]

In order to illustrate the possibility of observing squeezing phenomena for the pure and the coupled modes, we plot the right-hand sides of Eqs. (9), (10) and (12) as functions of $\kappa t$ in Figs. 2(a)–2(c), respectively. To illustrate the accuracy and validity of the analytic solutions reported here we show the variation of $(\Delta X_i)^2$ and $(\Delta Y_i)^2$ (where $i \in \{a, b, ab\}$) in Figs. 2(a)–2(c) using an exact numerical solution. For this, the time-dependent Schrödinger equation corresponding to Hamiltonian equation (1) is integrated by defining the operators as matrices. Figures 2(a)–2(c) clearly establish the correctness of our analytical results in the domain of interest.
as the exact numerical results are found to match perfectly with the approximate analytic results. To check the domain of validity of our analytic results in Fig. 2(d) we plot variations of $\langle \Delta X_a \rangle^2$ and $\langle \Delta Y_a \rangle^2$ with $\kappa$ for $t = 0.001$ s and keeping other parameters the same as in Figs. 2(a)–2(c). Interestingly, the analytic solution is found to be consistent with the exact numerical solution for a decent range of values of $\kappa$.

The plots in Figs. 2(a)–2(d) clearly show squeezing in all single modes and coupled mode. Figure 2(e) shows that the amount of squeezing can be controlled by controlling the value of atom-atom coupling constant $\kappa$.

Similarly, Fig. 2(f) illustrates that the amount of squeezing cannot be controlled by controlling $\epsilon$ as the increase in quadrature fluctuation with the increase in $\epsilon$ is negligibly small. The case of $\epsilon = 0$ implies no atomic tunneling and can be realized experimentally by imposing a large barrier between the two wells or having two separated wells. Figure 2(f) shows squeezing in this situation with similar time evolution characteristics as observed for nonzero values of $\epsilon$. However, no signature of other nonclassicalities (e.g., antibunching and entanglement) is found in this case. The absence of antibunching and entanglement for the case $\epsilon = 0$ can be observed from Figs. 3(d), 5(a), and 5(b) shown below.

A general feature in the figures is the oscillations with increasing amplitude at short time connected to the fact that no damping term is included in the Hamiltonian. These kinds of oscillations are generally seen in bosonic Hamiltonians without damping. The amplitude and period of the oscillations increase with the atom-atom coupling strength. The increase in period of oscillation with increase in $\kappa$ is negligibly small, but the increase in amplitude of oscillation (i.e., amount of squeezing) with the increase in $\kappa$ is considerable.

### IV. QUANTUM STATISTICS

To study the quantum statistical properties of the two-mode BEC system, we calculate the second-order correlation function for zero time delay

$$g^{(2)}(0) = \frac{\langle a^\dagger(t)a^\dagger(t)a(t)a(t) \rangle}{\langle a^\dagger(t)a(t) \rangle^2}.$$  \hspace{1cm} (13)

It is well known that if $0 \leq g^{(2)}(0) < 1$, then the corresponding particle number distribution is sub-Poissonian and is associated with the nonclassical phenomenon referred to as the antibunching effect. $g^{(2)}(0) = 1$ represents the coherent state with Poissonian state, while $g^{(2)}(0) > 1$ is the characteristics of the super-Poissonian distribution. Equation (13) can also be written in the form

$$g^{(2)}(0) - 1 = \frac{\langle \Delta N \rangle^2 - \langle N \rangle^2}{\langle N \rangle^2}.$$  \hspace{1cm} (14)

Here, the numerator $D = \langle \Delta N \rangle^2 - \langle N \rangle^2$ determines the quantum statistical properties as the denominator $\langle N \rangle^2$ is always positive. Precisely, $D < 0$, $D = 0$, and $D > 0$ correspond to sub-Poissonian, Poissonian, and super-Poissonian statistics, respectively. Now using $N_a = a^\dagger a$ and (3)–(6) we can obtain an analytic expression for $D_a = \langle \Delta N_a \rangle^2 - \langle N_a \rangle$ as

$$D_a = -|\alpha|^2[2 f_1 g_0 \alpha \beta^* + \text{c.c.}].$$  \hspace{1cm} (15)

In a similar manner we can obtain $D_b$ for mode $b$ as

$$D_b = -|\beta|^2[2 g_1 g_0 \beta \alpha^* + \text{c.c.}] = \frac{|\beta|^2}{|\alpha|^2} D_a.$$  \hspace{1cm} (16)

![FIG. 3. (Color online) Plot of $D_a$ versus $t$ in seconds for (a) pure mode $a$; (b) pure mode $b$; (c) couple mode $ab$ for $\kappa = 5$ Hz (smooth line), 8 Hz (dashed line), and 10 Hz (dot-dashed line) for $\epsilon = 500$ Hz, $\Delta \mu = 10^4$ Hz, and $\alpha = \beta = 5$; (d) $D_b$ with time $t$ using analytic solution for $\epsilon = 0$ Hz (smooth line), 250 Hz (dashed line), and 500 Hz (dot-dashed line) for $\kappa = 10$, $\Delta \mu = 10^4$ Hz, and $\alpha = \beta = 5$. Analytic expressions reported in this paper are used in obtaining (a)–(d).](image-url)
where we have used \( g_1 = f_1^* \) and \( g_2 = f_2^* \). In order to study the intermodal quantum statistics, we use the relevant second-order intermodal correlation function \( g^{(2)}_{ab}(0) \) for zero time delay as

\[
g^{(2)}_{ab}(0) = \frac{\langle a^{(t)} b^{(t)} (t) b^{(t)} (a(t)) \rangle}{\langle a^{(t)} (a(t)) \rangle \langle b^{(t)} (b(t)) \rangle}. \tag{17}
\]

The above equation can be alternatively written as

\[
g^{(2)}_{ab}(0) = 1 + \frac{D_{ab}}{\langle N_a \rangle \langle N_b \rangle}, \tag{18}
\]

where \( D_{ab} = \langle a^{(t)} b^{(t)} a b \rangle - \langle a^{(t)} a \rangle \langle b^{(t)} b \rangle \). Since the average number of the atom is positive, the sign of the numerator \( (D_{ab}) \) determines the quantum statistical properties. Now for the coupled mode \( ab \) the parameter \( D_{ab} \) is given by the equation

\[
D_{ab} = |\alpha|^2 + |\beta|^2 \{ f_1^* f_0 \alpha^2 \beta + c.c. |\}
\tag{19}
\]

In order to see whether the two-mode BEC system described above can show nonclassical (i.e., sub-Poissonian) photon statistics, we plot Eqs. (15), (16), and (19) in Fig. 3 with different values of \( \kappa \). From Figs. 3(a)(3(c), it is clear that the two-mode BEC shows antibunching for both \( a \) and \( b \) modes and also intermodal antibunching is observed for coupled mode \( ab \). Interestingly, with \( t \) the boson statistics is found to oscillate between classical and nonclassical regions in all three cases. It is also interesting to note that antibunching in the coupled mode \( ab \) is observed only when it is not present in pure modes \( a \) and \( b \). This interesting feature is also verified through exact numerical computation. This feature of antibunching can be analytically understood easily as we can use Eqs. (15), (16), and (19) to express \( D_{ab} \) as

\[
D_{ab} = -\frac{1}{2}(D_a + D_b) = -\frac{1}{2}(1 + |\beta|^2)D_a.
\]

As we have chosen \( D_a = D_b \) for plotting the figures, so \( D_{ab} = -D_a = -D_b \). This clearly explains the observation that \( D_{ab} \) is negative only when \( D_a \) and \( D_b \) are positive. It is easy to observe that this conclusion is valid even if \( |\alpha| \neq |\beta| \). To investigate the effect of \( \varepsilon \) on antibunching we plot the time evolution of \( D_a \) for various values of \( \varepsilon \) in Fig. 3(d). From Fig. 3(d) we can easily visualize that for mode \( a \) antibunching is absent for \( \varepsilon = 0 \) and the amount of antibunching increases with the increase in \( \varepsilon \).

V. INTERMODAL ENTANGLEMENT

In order to investigate the intermodal entanglement in atom-boson systems, we use three sufficient criteria for characterization of entanglement. The first two criteria, which we refer to as Hillary-Zubairy criterion-1 (HZ-1) and Hillary-Zubairy criterion-2 (HZ-2), were introduced by Hillary and Zubairy [42–44]; the third criterion is due to Duan et al. [45] and is usually referred to as the Duan criterion. There exist several sufficient inseparability criteria. However, we restrict ourselves to these three criteria as they have recently been found successful in detecting intermodal entanglement in other bosonic systems [21,28,39]. The first inseparability criterion of Hillary and Zubairy, i.e., HZ-1 criterion, is

\[
\langle N_a N_b \rangle < |\langle ab \rangle|^2. \tag{20}
\]

On the other hand, the HZ-2 criterion is given by

\[
\langle N_a \rangle \langle N_b \rangle < |\langle ab \rangle|^2. \tag{21}
\]

As all the above criteria for detecting intermodal entanglement are only sufficient (not necessary), a specific criterion may fail to identify entanglement detected by another criterion. Keeping this fact in mind, we use all these criteria to study the intermodal entanglement in two-mode BECs. This enhances the possibility of detection of entanglement and also helps us to compare the strength of these three criteria.

Let us first investigate the possibility of intermodal entanglement using the HZ-1 criterion. Using Eqs. (3)–(6) we obtain

\[
\langle N_a N_b \rangle - |\langle ab \rangle|^2 = |f_3|^2 |\alpha|^4 |\beta|^2 + |\alpha|^4 |\beta|^2 \\
+ \langle \{ f_1 f_2^{*} - f_2 f_1^{*} \} (|\alpha|^2 |\beta|^2 + |\alpha|^2 |\beta|^2) + c.c. \rangle. \tag{24}
\]

The negative value of the right-hand side of Eq. (24) gives us the signature of the intermodal entanglement. Now using the HZ-2 criterion, we obtain

\[
\langle N_a \rangle \langle N_b \rangle - |\langle ab \rangle|^2 = |f_3|^2 |\alpha|^4 |\beta|^2 + |\alpha|^4 |\beta|^2 \\
- \langle \{ g_1 g_2^{*} - g_2 g_1^{*} \} (|\alpha|^2 |\beta|^2 + |\alpha|^4 |\beta|^4) + c.c. \rangle. \tag{25}
\]

Using the Duan et al. criterion, we obtain

\[
d_{ab} = \langle (\Delta u)^2 \rangle + \langle (\Delta v)^2 \rangle - 2 \tag{26}
\]

In addition to these, we will also use Duan’s inseparability criterion [45]:

\[
(\langle \Delta u \rangle^2 + \langle \Delta v \rangle^2) - 2 < 0, \tag{22}
\]

where

\[
u = \frac{1}{\sqrt{2}} [a + a^*], \tag{23}
\]

\[
u = \frac{1}{\sqrt{2}} [a - a^*]. \tag{24}
\]

As all the above criteria for detecting intermodal entanglement are only sufficient (not necessary), a specific criterion may fail to identify entanglement detected by another criterion.
necessary, and on the other hand, they clearly indicate that the domain of nonclassicality detected by the HZ-2 criterion is larger than the same detected by the HZ-1 criterion for the specific choice of parameters that are used in Fig. 5. This observation is exactly opposite to what we have observed in Fig. 4. Thus it would be inappropriate to consider one of the Hillery-Zubairy criteria as being superior to the other. Further, the positive intercept at the \( Y \) axis by lines plotted in Fig. 5 clearly show that the Hillery-Zubairy criteria cannot detect any entanglement for \( \kappa = 0 \). We can also use Fig. 5 to conclude that for a specific value of \( \kappa \), depth of nonclassicality observed through Hillery-Zubairy criteria increases with the increase in \( \kappa \).

VI. HIGHER-ORDER NONCLASSICALITIES

In quantum optics higher-order nonclassical properties of radiation field (e.g., higher-order Hong-Mandel squeezing, higher-order antibunching, higher-order sub-Poissonian statistics, higher-order entanglement, etc.) are frequently studied ([20], and references therein). Initially most of the studies on higher-order nonclassicalities were limited to theoretical investigations. However, recently several experimental demonstrations [15–17] of higher-order nonclassicalities have been reported. Specifically, Allevi, Olivares, and Bondani [15] recently demonstrated higher-order nonclassicality in bipartite multimode states produced in a twin-beam experiment. They introduced a new criterion for higher-order nonclassicality and had shown that their higher-order nonclassical criterion is more useful in detecting weak nonclassicalities [15]. Earlier Pathak and Garcia [18] had theoretically shown that the depth of nonclassicality in higher-order antibunching increases with the order and thus higher-order nonclassical criterion introduced by them and other variants of higher-order antibunching criteria are more sensitive in the detection of weak nonclassicalities than their lower-order counterparts. This fact that higher-order nonclassicalities may be more useful in identifying the weak nonclassicalities and the recent success in experimental demonstration of higher-order nonclassicality have considerably enhanced the interest of the quantum optics community on the higher-order nonclassical properties of radiation fields. By contrast, in the analogous domain of BECs most of the recent studies on nonclassical characters of coupled BEC systems [27–34] are still limited to the investigation of lower-order nonclassicalities. These facts have motivated us to extend the BEC-light analogy and to study the higher-order nonclassical properties of two-mode BECs. Specifically, in the following sections we show that higher-order antibunching and higher-order entanglement can be seen in two-mode BECs.

A. Higher-order antibunching

The notion of higher-order antibunching was introduced by Lee in 1990 [46]. Initially it was thought to be a rare phenomenon, but in 2006 it was shown by some of the present authors that it is not really a rare phenomenon [19]; since then it has been observed in several quantum optical systems ([20], and references therein). Signature of this higher-order nonclassicality can be obtained through any of a set of equivalent but different criteria, all of which can be viewed as modified Lee criterion [46]. In order to investigate \( (n-1) \)th order antibunching in two-mode BECs we use the following criterion of Pathak and Garcia [18]:

\[
\langle a^\dagger a^n \rangle - \langle a^\dagger a \rangle^n < 0. \quad (27)
\]
Here \( n = 2 \) corresponds to the usual antibunching, and if the above criterion is satisfied by a quantum state for \( n \geq 3 \), then the quantum state is referred to as higher-order antibunched. Using Eqs. (3)–(6) and (27) we obtain

\[
\langle a^\dagger a^n \rangle - \langle a^\dagger a \rangle^n = |f_3|^2 n(n-1)(n-2) |\alpha|^{2(n+1)} \\
- \{ f_1 f_2 n(n-1) |\alpha|^{2(n-1)} \alpha^\ast \beta + c.c. \}.
\]

Similarly, for mode \( b \) we obtain

\[
\langle b^\dagger b^n \rangle - \langle b^\dagger b \rangle^n = |g_3|^2 n(n-1)(n-2) |\beta|^{2(n+1)} \\
- \{ g_1 g_2 n(n-1) |\beta|^{2(n-1)} \beta^\ast \alpha + c.c. \}.
\]

To illustrate that it is possible to observe higher-order antibunching in two-mode BEC systems, we have plotted Eqs. (28) and (29) in Fig. 6. Negative regions of the plots clearly illustrate that the individual modes of two-mode BEC systems are in a higher-order antibunched state. However, with the rescaled time \( \kappa t \) the quantum states of the individual modes oscillate between classical region and nonclassical region with respect to this criterion. It is also clear from Figs. 6(a) and 6(b) that the amount of antibunching increases with the order. This observation is consistent with analogous observations reported in the context of quantum optical systems [15,19].

### B. Higher-order entanglement

In the same spirit as followed in the previous section, we may now investigate the possibility of observing higher-order intermodal entanglement in two-mode BECs. In order to do that we use the Hillery-Zubairy criterion of higher-order entanglement [42]. According to this criterion, a state is entangled if

\[
E_{a,n,m} = \langle (a^\dagger)^m a^n b^m \rangle - |\langle a^n (b^m) \rangle|^2 < 0. \tag{30}
\]

As \( m \) and \( n \) are nonzero positive integers, the lowest possible values of \( m \) and \( n \) are \( m = n = 1 \). Clearly for this lowest possible value, the HZ criterion for higher-order entanglement (30) reduces to the usual HZ-1 criterion (20). Thus with respect to this criterion a quantum state will be referred to as a higher-order entangled state if it satisfies (30) for any choice of integers \( m \) and \( n \) satisfying \( m + n \geq 3 \). In the present study, we use \( m = n \). Now using (3)–(6) and (30) we obtain

\[
E_{a,n} = \langle (a^\dagger)^n a^n b^n \rangle - |\langle a^n b^n \rangle|^2 = |f_3|^2 n^2 (|\alpha|^2 n |\beta|^{2(n+1)} + |\alpha|^2 |\beta|^{2(n+1)}) \\
+ \left( [f_1 f_2 - f_1 f_2] n |\alpha|^2 |\beta|^{2(n-1)} + c.c. \right). \tag{31}
\]

As the negative values of the right-hand side (RHS) of (31) are the signatures of the higher-order entanglement (for \( n \geq 2 \) in this case), we plot the RHS of (31) in Fig. 7 for \( n = 1, 2, \) and \( 3 \). Negative regions of the plot for \( n = 2 \) and \( n = 3 \) are depicting the presence of higher-order entanglement in two-mode BECs. Inclusion of the plot for \( n = 1 \) (which corresponds to usual entanglement observed through the HZ-1 criterion) helps us to conclude that the depth of nonclassicality is more in the case of higher-order nonclassicality. Thus this particular higher-order criterion of entanglement is expected to be more sensitive and capable of detecting weak nonclassicality from experimental data. However, if this sufficient criterion fails to detect an entanglement for \( n = 1 \), then this criterion would also fail to detect that entanglement for all other values of \( n \). This is so because all the lines present in Fig. 7 intercept the \( X \) axis at the same points.

![FIG. 6. (Color online) Plot of higher-order antibunching for (a) mode \( a \) for \( n = 2 \) (smooth line), \( n = 3 \) (dashed line), and \( n = 4 \) (dot-dashed line); (b) mode \( b \) for \( n = 2 \) (smooth line), \( n = 3 \) (dashed line), and \( n = 4 \) (dot-dashed line). Here, \( \alpha = \beta = 1 \), \( \kappa = 10 \) Hz, \( \frac{\delta}{\Delta} = 50 \), and \( \Delta \mu = 10^5 \) Hz. Negative parts of the plot for \( n = 2 \) depict usual antibunching as shown in Fig. 3. Negative regions of the plot for \( n = 3, 4 \) shows the existence of higher-order antibunching.](033628-8)

![FIG. 7. (Color online) Plot of higher-order entanglement for \( n = 1 \) (solid line), \( n = 2 \) (dashed line), and \( n = 3 \) (dot-dashed line). Here \( \alpha = \beta = 1 \), \( \kappa = 10 \) Hz, \( \frac{\delta}{\Delta} = 50 \), and \( \Delta \mu = 10^5 \) Hz.](033628-8)
VII. CONCLUSIONS

The existence of a light-BEC analogy is well known. The analogy enabled us to apply some methods developed earlier for the study of quantum optical systems to systematically study the nonclassical character of two-mode BECs. Various types of lower-order and higher-order nonclassicalities are observed. In particular, we have shown that a two-mode coupled BEC system described by the generalized Hamiltonian (1) is an excellent example of a nonclassical system, where single-mode and intermodal squeezing, intermodal antibunching, and intermodal entanglement can be observed in this physical system. Furthermore, we found the signatures of higher-order nonclassical characters of two-mode BEC systems by showing the possibility of higher-order antibunching and higher-order entanglement. It is observed that the depth of nonclassicality is higher if the order is higher. In the case of entanglement we have used a set of inseparability criteria each of which is sufficient but not essential [39]. Interestingly, it is observed that the Duan et al. criterion (22) could not characterize entanglement in the present system. However, HZ-1 (20) and HZ-2 (21) criteria succeed to do so. Also, it is observed that the HZ-2 (HZ-1) criterion was successful in detecting entanglement in some regions where entanglement was not detected by the HZ-1 (HZ-2) criterion. The mutual relation among the observed nonclassicalities is discussed and their evolution (variation) with time ($t$ or rescaled-time $\kappa t$) and the ratio of the single boson tunneling amplitude ($\kappa$) and the coupling constant for the intramodal interaction ($\epsilon$) are also shown. The study indicates that the amount of quantumness (nonclassicality) may be controlled by controlling $\epsilon$ and $\kappa$. The procedure adopted here is quite general and may also be used to study the nonclassical characters of coupled BEC systems described by Hamiltonians that are not equivalent to (1). Recent reports on applications of BEC-based systems in quantum information processing, experimental demonstrations of higher-order nonclassicality, and frequent realizations of two-mode BEC systems indicate a strong possibility for experimental verification of the present work. It also indicates the results presented here may find applications in quantum information processing.

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