CONTACT CR-WARPED PRODUCT SUBMANIFOLDS OF NEARLY TRANS-SASAKIAN MANIFOLDS

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Abstract. Recently, many authors studied the relations between the squared norm of the second fundamental form (extrinsic invariant) and the warping function (intrinsic invariant) for warped product submanifolds (see [1, 7, 14]). Inspired by those relations we establish a general sharp inequality, namely \[ \|h\|^2 \geq 2s\|\nabla \ln f\|^2 + \alpha^2 - \beta^2, \]
for contact CR-warped products of nearly trans-Sasakian manifolds. Our inequality generalizes all derived inequalities for contact CR-warped products either in any contact metric manifold. The equality case is also handled.

1. INTRODUCTION

A \((2n + 1)\)-dimensional differentiable manifold \(\tilde{M}\) of class \(C^\infty\) is said to have a contact structure (J.W. Gray [13]) if the structural group of its tangent bundle reduces to \(U(n) \times 1\); equivalently (Sasaki and S. Hatakeyama [19]), an almost contact structure is given by a triple \((\phi, \xi, \eta)\) satisfying certain conditions. Many different types of almost contact structures are defined in the literature like cosymplectic, Sasakian, quasi-Sasakian, normal contact, Kenmotsu, trans-Sasakian. These type of structures bear sufficient resemblance to cosymplectic and Sasakian structures. Later on, D. Chinea and C. Gonzalez generalized these structures into a general system. They divided almost contact metric manifolds into twelve well known classes. An almost contact metric manifold is nearly trans-Sasakian if it belongs to the class \(C_1 \oplus C_5 \oplus C_6\). Recently, C. Gherghe introduced a nearly trans-Sasakian structure of type \((\alpha, \beta)\), which generalizes trans-Sasakian structure in the same sense as nearly Sasakian generalizes Sasakian ones. Moreover, a nearly trans-Sasakian of type \((\alpha, \beta)\) is nearly-Sasakian [4] or nearly...
Kenmotsu [20] or nearly cosymplectic [3] according to $\beta = 0$ or $\alpha = 0$ or $\alpha = \beta = 0$, respectively.

On the other hand, the idea of warped product submanifolds was introduced by Chen in [7] (see also [6, 8]). He studied the warped product CR-submanifolds of a Kaehler manifold. He proved many interesting results on the existence of warped products and established general sharp inequalities for the second fundamental form in terms of the warping function $f$. Later on, many articles have been appeared for the same inequalities in almost Hermitian as well as almost contact metric manifolds (see [1, 5, 17]). In this paper, we study the warped product contact CR-submanifolds of nearly trans-Sasakian manifolds. In the beginning, we prove some existence and non-existence results and then obtain a general sharp inequality for the second fundamental form in terms of the warping function $f$ and the smooth functions $\alpha$, $\beta$ on a nearly trans-Sasakian manifold. The inequality obtained in this paper is more general as it generalizes all inequalities obtained for contact CR-warped products in contact metric manifolds.

2. Preliminaries

A $(2n+1)-$dimensional $C^\infty$ manifold $\tilde{M}$ is said to have an almost contact structure if there exist on $\tilde{M}$ a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1–form $\eta$ satisfying [3]

\[(\nabla_X\phi)Y = -I + \eta \otimes \xi, \ \phi\xi = 0, \ \eta \circ \phi = 0, \ \eta(\xi) = 1.\]

There always exists a Riemannian metric $g$ on an almost contact manifold $\tilde{M}$ satisfying the following compatibility condition

\[\eta(X) = g(X, \xi), \ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\]

where $X$ and $Y$ are vector fields on $\tilde{M}$.

There are two known classes of almost contact metric manifolds, namely Sasakian and Kenmotsu manifolds. Sasakian manifolds are characterized by the tensorial relation $(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X$, while the Kenmotsu manifolds are given by the tensor equation $(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$.

An almost contact metric structure $(\phi, \xi, \eta, g)$ on $\tilde{M}$ is called a trans-Sasakian structure [18] if $(\tilde{M} \times R, J, G)$ belongs to the class $W_4$ of the Gray-Hervella classification of almost Hermitian manifolds [12], where $J$ is the almost complex structure on $\tilde{M} \times R$ defined by $J(X, ad/dt) = (\phi X - a\xi, \eta(X)d/dt)$, for all vector fields $X$ on $\tilde{M}$ and smooth functions $a$ on $M \times R$ and $G$ is the product metric on $M \times R$. This may be expressed by the condition

\[(\nabla_X\phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)\]
for some smooth functions $\alpha$ and $\beta$ on $\bar{M}$, and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$.

An almost contact metric structure $(\phi, \xi, \eta, g)$ on $\bar{M}$ is called a nearly trans-Sasakian structure [11] if

$$
(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y).
$$

Moreover, a nearly trans-Sasakian of type $(\alpha, \beta)$ is nearly-Sasakian [4] or nearly Kenmotsu [20] or nearly cosymplectic [3] according as $\beta = 0$, $\alpha = 1$; or $\alpha = 0$, $\beta = 1$; or $\alpha = \beta = 0$, respectively.

The covariant derivative of the tensor field $\phi$ is defined as

$$
(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y.
$$

Let $M$ be submanifold of an almost contact metric manifold $\bar{M}$ with induced metric $g$ and let $\nabla$ and $\nabla^\perp$ be the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on $M$ and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of a vector bundle $TM$ over $M$, then the Gauss and Weingarten formulae are given by

$$
\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)
$$

and

$$
\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N,
$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $h$ and $A_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field $N$) respectively for the immersion of $M$ into $\bar{M}$. They are related as

$$
g(h(X, Y), N) = g(A_N X, Y),
$$

where $g$ denotes the Riemannian metric on $\bar{M}$ as well as induced on $M$.

Bishop and O’Neill [2] introduced the notion of warped product manifolds. They defined these manifolds as: Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannian manifolds and $f$ be a positive differentiable function on $N_1$. The warped product of $N_1$ and $N_2$ is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$
g = g_1 + f^2 g_2.
$$

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function $f$ is constant.

We recall the following general result for later use.

**Lemma 2.1.** ([2]). Let $M = N_1 \times_f N_2$ be a warped product manifold with the warping function $f$, then
\( \nabla_X Y \in \Gamma(\mathcal{T}N_1) \) is the lift of \( \nabla_X Y \) on \( N_1 \),

(ii) \( \nabla_X Z = \nabla_Z X = (X \ln f)Z \),

(iii) \( \nabla_Z W = \nabla^{N_2}_Z W - g(Z, W) \nabla \ln f \)

for each \( X, Y \in \Gamma(\mathcal{T}N_1) \) and \( Z, W \in \Gamma(\mathcal{T}N_2) \), where \( \nabla \ln f \) is the gradient of \( \ln f \) and \( \nabla \) and \( \nabla^{N_2} \) denote the Levi-Civita connections on \( M \) and \( N_2 \), respectively.

For a Riemannian manifold \( M \) of dimension \( n \) and a smooth function \( f \) on \( M \), we recall \( \nabla f \), the gradient of \( f \) which is defined by

\[
(2.9) \quad g(\nabla f, X) = X(f),
\]

for any \( X \in \Gamma(TM) \). As a consequence, we have

\[
(2.10) \quad \| \nabla f \|^2 = \sum_{i=1}^{n} (e_i(f))^2
\]

for an orthonormal frame \( \{e_1, \cdots, e_n\} \) on \( M \).

3. Contact CR-Warped Product Submanifolds

In this section first we recall the invariant, anti-invariant and contact CR-submanifolds. For submanifolds tangent to the structure vector field \( \xi \), there are different classes of submanifolds. We mention the following:

(i) A submanifold \( M \) tangent to \( \xi \) is an invariant submanifold if \( \phi \) preserves any tangent space of \( M \), that is, \( \phi(T_pM) \subset T_pM \), for every \( p \in M \).

(ii) A submanifold \( M \) tangent to \( \xi \) is an anti-invariant submanifold if \( \phi \) maps any tangent space of \( M \) into the normal space , that is, \( \phi(T_pM) \subset T^\perp_p M \), for every \( p \in M \).

Let \( M \) be a Riemannian manifold isometrically immersed in an almost contact metric manifold \( \bar{M} \), then for every \( p \in M \) there exists a maximal invariant subspace denoted by \( D_p \) of the tangent space \( T_pM \) of \( M \). If the dimension of \( D_p \) is same for all values of \( p \in M \), then \( D_p \) gives an invariant distribution \( D \) on \( M \).

A submanifold \( M \) of an almost contact manifold \( \bar{M} \) is said to be a contact CR-submanifold if there exists on \( M \) a differentiable distribution \( D \) whose orthogonal complementary distribution \( D^\perp \) is anti-invariant, that is;

(i) \( TM = D \oplus D^\perp \oplus \langle \xi \rangle \)

(ii) \( D \) is an invariant distribution, i.e., \( \phi D \subset TM \)

(iii) \( D^\perp \) is an anti-invariant distribution, i.e., \( \phi D^\perp \subset T^\perp M \).
A contact CR-submanifold is anti-invariant if $D_p = \{0\}$ and invariant if $D^\perp_p = \{0\}$ respectively, for every $p \in M$. It is a proper contact CR-submanifold if neither $D_p = \{0\}$ nor $D^\perp_p = \{0\}$, for each $p \in M$.

If $\nu$ is the $\phi$–invariant subspace of the normal bundle $T^\perp M$, then in case of contact CR-submanifold, the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = \phi D^\perp \oplus \nu,$$

where $\nu$ is the $\phi$–invariant normal subbundle of $T^\perp M$.

In this section, we investigate the warped products $M = N^\perp \times f N_T$ and $M = N_T \times f N^\perp$, where $N_T$ and $N^\perp$ are invariant and anti-invariant submanifolds of a nearly trans-Sasakian manifold $\bar{M}$, respectively. First we discuss the warped products $M = N^\perp \times f N_T$, here two possible cases arise:

(i) $\xi$ is tangent to $N_T$,
(ii) $\xi$ is tangent to $N^\perp$.

We start with the case (i).

**Theorem 3.1.** Let $\bar{M}$ be a nearly trans-Sasakian manifold. Then there do not exist warped product submanifolds $M = N^\perp \times f N_T$ such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N^\perp$ is an anti-invariant submanifold of $\bar{M}$, unless $\bar{M}$ is nearly $\alpha$–Sasakian.

**Proof.** Consider $\xi \in \Gamma(TN_T)$ and $Z \in \Gamma(TN^\perp)$, then by the structure equation of nearly trans-Sasakian, we have $(\bar{\nabla}_Z \phi)\xi + (\bar{\nabla}_\xi \phi)Z = -\alpha Z - \beta \phi Z$. Using (2.4), we obtain $-\phi \bar{\nabla}_Z \xi + \nabla_\xi \phi Z - \phi \bar{\nabla}_\xi Z = -\alpha Z - \beta \phi Z$. Then from Lemma 2.1(ii) and (2.5), we derive

$$\nabla_\xi \phi Z - 2\phi h(Z, \xi) = -\alpha Z - \beta \phi Z.$$

Taking the inner product with $\phi Z$ in (3.2) and then using (2.2) and the fact that $\xi \in \Gamma(TN_T)$, we get $\beta \|Z\|^2 = 0$, for non zero function smooth function $\beta$ on $\bar{M}$ and hence we conclude that $M$ is invariant, which proves the theorem.

Now, we will discuss the other case, when $\xi$ is tangent to $N^\perp$.

**Theorem 3.2.** Let $\bar{M}$ be a nearly trans-Sasakian manifold. Then there do not exist warped product submanifolds $M = N^\perp \times f N_T$ such that $N^\perp$ is an anti-invariant submanifold tangent to $\xi$ and $N_T$ is an invariant submanifold of $\bar{M}$, unless $\bar{M}$ is nearly $\beta$–Kenmotsu.

**Proof.** Consider $\xi \in \Gamma(TN^\perp)$ and $X \in \Gamma(TN_T)$, then we have$(\bar{\nabla}_X \phi)\xi + (\bar{\nabla}_\xi \phi)X = -\alpha X - \beta \phi X$. Using (2.4), we get

$$-\phi \bar{\nabla}_X \xi + \bar{\nabla}_\xi \phi X - \phi \bar{\nabla}_\xi X = -\alpha X - \beta \phi X.$$
Taking the inner product with $X$ in (3.3) and using (2.2), (2.5), Lemma 2.1 (ii) and the fact that $\xi$ is tangent to $N_\perp$, we obtain $\alpha \|X\|^2 = 0$, for some smooth function $\alpha$ on $\bar{M}$. Thus, we conclude that $M$ is anti-invariant submanifold of a nearly trans-Sasakian manifold $\bar{M}$ otherwise $\bar{M}$ is nearly $\beta$–Kenmotsu. This completes the proof.

Now, we will discuss the warped product $M = N_T \times_f N_\perp$ such that the structure vector field $\xi$ is tangent to $N_\perp$.

**Theorem 3.3.** Let $\bar{M}$ be a nearly trans-Sasakian manifold. Then there do not exist the warped product submanifolds $M = N_T \times_f N_\perp$ such that $N_\perp$ is an anti-invariant submanifold tangent to $\xi$ and $N_T$ an invariant submanifold of $\bar{M}$.

**Proof.** If we consider $X \in \Gamma(TN_T)$ and the structure vector field $\xi$ is tangent to $N_\perp$, then by (2.3), we have $(\bar{\nabla}_X \phi)\xi + (\bar{\nabla}_\xi \phi)X = -\alpha X - \beta \phi X$. Using (2.4), we obtain $\bar{\nabla}_\xi \phi X - \phi \bar{\nabla}_X \xi - \phi \bar{\nabla}_\xi X = -\alpha X - \beta \phi X$. Then by (2.5) and Lemma 2.1 (ii), we derive

$$(3.4) \quad 2(\xi \ln f)\phi Z + 2\phi h(Z, \xi) - \bar{\nabla}_\xi \phi Z = \alpha Z + \beta \phi Z.$$ 

Hence, the result is obtained by taking the inner product with $\xi$ in (3.4).

If we consider the structure vector field $\xi$ tangent to $N_T$ for the warped product $M = N_T \times_f N_\perp$, then we prove the following result for later use.

**Lemma 3.1.** Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a nearly trans-Sasakian manifold $\bar{M}$ such that $N_T$ and $N_\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. Then, we have

(i) $\xi(\ln f) = \beta$,

(ii) $g(h(X, Y), \phi Z) = 0$,

(iii) $g(h(X, W), \phi Z) = g(h(X, Z), \phi W) = -\{h(\xi \ln f) + \alpha \eta(X)\}g(Z, W)$, 

(iv) $g(h(\xi, Z), \phi W) = -\alpha g(Z, W)$

for every $X, Y \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_\perp)$.

**Proof.** If $\xi$ is tangent to $N_T$, then for any $Z \in \Gamma(TN_\perp)$, we have $(\bar{\nabla}_\xi \phi)Z + (\bar{\nabla}_Z \phi)\xi = -\alpha Z - \beta \phi Z$. Then from (2.4), (2.5) and Lemma 2.1 (ii), we obtain

$$(3.5) \quad 2(\xi \ln f)\phi Z + 2\phi h(Z, \xi) - \bar{\nabla}_\xi \phi Z = \alpha Z + \beta \phi Z.$$ 

Taking the inner product with $\phi Z$ in (3.5) and using (2.2), we derive

$$(3.6) \quad 2(\xi \ln f)\|Z\|^2 - g(\bar{\nabla}_\xi \phi Z, \phi Z) = \beta \|Z\|^2.$$
On the other hand, by the property of Riemannian connection, we have \( \xi g(\phi Z, \phi Z) = 2g(\nabla_\xi \phi Z, \phi Z) \). By (2.2) and the property of Riemannian connection, we get
\[
(3.7) \quad g(\nabla_\xi Z, Z) = g(\nabla_\xi \phi Z, \phi Z).
\]
Using this fact in (3.6) and then from (2.5) and Lemma 2.1 (ii), we deduce that
\[
(3.8) \quad g(\nabla_\xi Z, Z) = g(h(X, Z), \phi W) - 2g(h(X, Z), \phi Z) = \alpha \eta(X) g(Z, W)
\]
Thus, the second part can be obtained by taking the inner product in (3.8) with \( Y \), for any \( Y \in \Gamma(TN_T) \). Again, taking the inner product in (3.8) with \( W \) for any \( W \in \Gamma(TN_{\perp}) \), we get
\[
\alpha \eta(X) g(Z, W) = g(h(X, Z), \phi W) - 2g(h(X, Z), \phi Z).
\]
By polarization identity, we get
\[
\alpha \eta(X) g(Z, W) = g(h(X, Z), \phi W) - (\phi X \ln f) g(Z, W)
\]
Then from (3.9) and (3.10), we obtain
\[
(3.11) \quad g(h(X, Z), \phi W) = g(h(X, W), \phi Z),
\]
which is the first equality of \((iii)\). Using (3.11) either in (3.9) or in (3.10), we get the second equality of \((iii)\). Now, for the last part, replacing \( X \) by \( \xi \) in the third part of this lemma. This proves the lemma completely.

Now, we have the following characterization theorem.

**Theorem 3.4.** Let \( M \) be a contact CR-submanifold of a nearly trans-Sasakian manifold \( \bar{M} \) with integrable invariant and anti-invariant distribution \( D \oplus \langle \xi \rangle \) and \( D_{\perp} \). Then \( M \) is locally a contact CR-warped product if and only if the shape operator of \( M \) satisfies
\[
(3.12) \quad A_{\phi W} X = -\phi X \mu W - \alpha \eta(X) W, \quad X \in \Gamma(D \oplus \langle \xi \rangle), \ W \in \Gamma(D_{\perp})
\]
for some smooth function \( \mu \) on \( M \) satisfying \( V(\mu) = 0 \) for every \( V \in \Gamma(D_{\perp}) \).

**Proof.** Direct part follows from the Lemma 3.1 \((iii)\). For the converse, suppose that \( M \) is contact CR-submanifold satisfying (3.12), then we have \( g(h(X, Y) \phi W) = \)}
\( g(A_{\phi W}X, Y) = 0 \), for any \( X, Y \in \Gamma(D \oplus \langle \xi \rangle) \) and \( W \in \Gamma(D^\perp) \). Using (2.2) and (2.5), we get \( g(\nabla_X Y, \phi W) = -g(\phi \nabla_X Y, W) = 0 \). Then from (2.4), we obtain

\[
(3.13) \quad g((\nabla_X \phi)Y, W) = g(\nabla_X \phi Y, W).
\]

Similarly, we have

\[
(3.14) \quad g((\nabla_Y \phi)X, W) = g(\nabla_Y \phi X, W).
\]

Then from (3.13) and (3.14), we derive

\[
(3.15) \quad g((\nabla_X \phi)Y + (\nabla_Y \phi)X, W) = g(\nabla_X \phi Y + \nabla_Y \phi X, W).
\]

Using (2.3) and the fact that \( \xi \) is tangent to \( N_T \), then by orthogonality of two distributions, we obtain

\[
(3.16) \quad g(\nabla_X \phi Y + \nabla_Y \phi X, W) = 0.
\]

This means that \( \nabla_X \phi Y + \nabla_Y \phi X \in \Gamma(D \oplus \langle \xi \rangle) \), for any \( X, Y \in \Gamma(D \oplus \langle \xi \rangle) \), that is \( D \oplus \langle \xi \rangle \) is integrable and its leaves are totally geodesic in \( M \). So far as the anti-invariant distribution \( D^\perp \) is concerned, it is integrable on \( M \) (cf. [16], Theorem 8.1). Let \( N_\perp \) be the leaf of \( D^\perp \) and \( h^* \) be the second fundamental form of \( N_\perp \) in \( M \). Then for any \( X \in \Gamma(D \oplus \langle \xi \rangle) \) and \( Z, W \in \Gamma(D^\perp) \), we have \( g(h^*(Z, W), \phi X) = g(\nabla_Z W, \phi X) \). Using (2.2), (2.4) and (2.5), we obtain \( g(h^*(Z, W), \phi X) = g((\nabla_Z \phi)W, X) - g(\nabla_Z \phi W, X) \). Then from (2.6) and (2.7), we get

\[
(3.17) \quad g(h^*(Z, W), \phi X) = g((\nabla_Z \phi)W, X) + g(A_{\phi W}X, Z).
\]

Using (3.12), we derive

\[
(3.18) \quad g(h^*(Z, W), \phi X) = g((\nabla_Z \phi)W, X) + \{(\phi X)\mu - \alpha\eta(X)\}g(Z, W).
\]

Similarly, we obtain

\[
(3.19) \quad g(h^*(Z, W), \phi X) = g((\nabla_W \phi)Z, X) + \{(\phi X)\mu - \alpha\eta(X)\}g(Z, W).
\]

Then from (3.18) and (3.19), we get

\[
(3.20) \quad 2g(h^*(Z, W), \phi X) = g((\nabla_Z \phi)W + (\nabla_W \phi)Z, X)
+ 2\{(\phi X)\mu - \alpha\eta(X)\}g(Z, W).
\]

Using the structure equation of nearly trans-Sasakian and the fact that \( \xi \) is tangent to \( N_T \), we obtain

\[
(3.21) \quad 2g(h^*(Z, W), \phi X) = 2\alpha g(Z, W)g(\xi, X) + 2\{(\phi X)\mu - \alpha\eta(X)\}g(Z, W).
\]
That is
\[(3.22) \quad g(h^*(Z, W), \phi X) = (\phi X)\mu g(Z, W).\]

Using (2.9), we derive
\[(3.23) \quad g(h^*(Z, W), \phi X) = g(\nabla \mu, \phi X)g(Z, W).\]

From the last relation, we obtain that
\[(3.24) \quad h^*(Z, W) = (\nabla \mu)g(Z, W).\]

The above relation shows that the leaves of $D^\perp$ are totally umbilical in $M$ with mean curvature vector $\nabla \mu$. Moreover, the condition $V\mu = 0$, for any $V \in \Gamma(D^\perp)$ implies that the leaves of $D^\perp$ are extrinsic spheres in $M$, that is the integral manifold $N_\perp$ of $D^\perp$ is umbilical and its mean curvature vector field is non zero and parallel along $N_\perp$. Hence, by a result of [15] $M$ is locally a warped product $M = N_T \times f N_\perp$, where $N_T$ and $N_\perp$ denote the integral manifolds of the distributions $D \oplus \langle \xi \rangle$ and $D^\perp$, respectively and $f$ is the warping function. Thus, the theorem is proved completely. 

**4. INEQUALITY FOR CONTACT CR-WARPED PRODUCTS**

In the following section we obtain a general sharp inequality for the length of second fundamental form of warped product submanifold. We prove the following main result of this section.

**Theorem 4.1.** Let $M = N_T \times f N_\perp$ be a contact CR-warped product submanifold of a nearly trans-Sasakian manifold $\bar{M}$ such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ an anti-invariant submanifold of $\bar{M}$. Then, we have

(i) The second fundamental form of $M$ satisfies the inequality
\[(4.1) \quad ||h||^2 \geq 2s||\nabla \ln f||^2 + \alpha^2 - \beta^2\]

where $s$ is the dimension of $N_\perp$ and $\nabla \ln f$ is the gradient of $\ln f$.

(ii) If the equality sign of (4.1) holds identically, then $N_T$ is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\bar{M}$. Moreover, $M$ is a minimal submanifold in $\bar{M}$.

**Proof.** Let $\bar{M}$ be a $(2n + 1)$–dimensional nearly trans-Sasakian manifold and $M = N_T \times f N_\perp$ be an $m$–dimensional contact CR-warped product submanifolds of $\bar{M}$. Let us consider dim $N_T = 2p + 1$ and dim $N_\perp = s$, then $m = 2p + 1 + s$. Let $\{e_1, \ldots, e_p, \phi e_1 = e_{p+1}, \ldots, \phi e_p = e_{2p}, e_{2p+1} = \xi\}$ and $\{e_{(2p+1)+1}, \ldots, e_m\}$ be the local orthonormal frames on $N_T$ and $N_\perp$, respectively. Then the orthonormal
frames in the normal bundle $T^\perp M$ of $\phi D^\perp$ and $\nu$ are \{\phi e_{(2p+1)+1}, \ldots, \phi e_m\} and \{e_{m+s+1}, \ldots, e_{2n+1}\}, respectively. Then the length of second fundamental form $h$ is defined as

\[(4.2) \quad \|h\|^2 = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^m g(h(e_i, e_j), e_r)^2.\]

For the assumed frames, the above equation can be written as

\[(4.3) \quad \|h\|^2 = \sum_{r=m+1}^{m+s} \sum_{i,j=1}^m g(h(e_i, e_j), e_r)^2 + \sum_{r=m+s+1}^{2n+1} \sum_{i,j=1}^m g(h(e_i, e_j), e_r)^2.\]

The first term in the right hand side of the above equality is the $\phi D^\perp$—component and the second term is $\nu$—component. If we equate only the $\phi D^\perp$—component, then we have

\[(4.4) \quad \|h\|^2 \geq \sum_{r=m+1}^{m+s} \sum_{i,j=1}^m g(h(e_i, e_j), e_r)^2.\]

For the given frame of $\phi D^\perp$, the above equation will be

\[\|h\|^2 \geq \sum_{k=(2p+1)+1}^{m} \sum_{i,j=1}^m g(h(e_i, e_j), \phi e_k)^2.\]

Let us decompose the above equation in terms of the components of $h(D, D)$, $h(D, D^\perp)$ and $h(D^\perp, D^\perp)$, then we have

\[(4.5) \quad \|h\|^2 \geq \sum_{k=2p+2}^{m} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \phi e_k)^2 + 2 \sum_{k=2p+2}^{m} \sum_{i=1}^{m} \sum_{j=2p+2}^{2p+1} g(h(e_i, e_j), \phi e_k)^2 + \sum_{k=2p+2}^{m} \sum_{i,j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2.\]

By Lemma 3.1 (ii), the first term of the right hand side of (4.5) is identically zero and we shall compute the next term and will left the last term

\[\|h\|^2 \geq 2 \sum_{k=2p+2}^{m} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2.\]

As $j, k = 2p + 2, \ldots, m$, then the above equation can be written for one summation as
\[ \| h \|^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2. \]

Making use of Lemma 3.1 (iii), the above inequality will be

\[ (4.6) \quad \| h \|^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f + \alpha \eta(e_i))^2 g(e_j, e_k)^2. \]

The above expression can be written as

\[ (4.7) \quad \| h \|^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)^2 g(e_j, e_k)^2 + 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\alpha \eta(e_i))^2 g(e_j, e_k)^2 + 4 \alpha \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f) \eta(e_i) g(e_j, e_k)^2. \]

The last term of (4.7) is identically zero for the given frames. Thus, the above relation gives

\[ (4.8) \quad \| h \|^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)^2 g(e_j, e_k)^2 + 2 \alpha^2 s. \]

On the other hand, from (2.10), we have

\[ (4.9) \quad \| \nabla \ln f \|^2 = \sum_{i=1}^{p} (e_i \ln f)^2 + \sum_{i=1}^{p} (\phi e_i \ln f)^2 + (\xi \ln f)^2. \]

Now, the equation (4.8) can be modified as

\[ \| h \|^2 \geq 2 \sum_{i=1}^{2p} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)^2 g(e_j, e_k)^2 + 2 \sum_{j,k=2p+1}^{m} (\xi \ln f)^2 g(e_j, e_k)^2 + 2 \alpha^2 s - 2 \sum_{j,k=2p+1}^{m} (\xi \ln f)^2 g(e_j, e_k)^2, \]

or

\[ \| h \|^2 \geq 2 \alpha^2 s - 2 \sum_{j,k=2p+2}^{m} (\xi \ln f)^2 g(e_j, e_k)^2. \]
\[ +2 \sum_{i=1}^{p} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)^2 g(e_j, e_k)^2 \]
\[ +2 \sum_{i=1}^{p} \sum_{j,k=2p+2}^{m} (e_i \ln f)^2 g(e_j, e_k)^2 \]
\[ +2 \sum_{j,k=2p+2}^{m} (\xi \ln f)^2 g(e_j, e_k)^2, \] (since $\phi \xi \ln f = 0$).

Therefore, using Lemma 3.1 (i) and (4.9), we arrive at
\[ \|h\|^2 \geq 2s\alpha^2 - 2s\beta^2 + 2s\|\nabla \ln f\|^2, \]
which is the inequality (4.1). Let $h^*$ be the second fundamental form of $N_\perp$ in $M$, then from (3.24), we have
\[ h^*(Z, W) = g(Z, W)\nabla \ln f, \]
for any $Z, W \in \Gamma(D^\perp)$. Now, assume that the equality case of (4.1) holds identically. Then from (4.3), (4.5) and (4.7), we obtain
\[ h(D, D) = 0, \ h(D^\perp, D^\perp) = 0, \ h(D, D^\perp) \subset \phi D^\perp. \]

Since $N_T$ is a totally geodesic submanifold in $M$ (by Lemma 2.1 (i)), using this fact with the first condition in (4.11) implies that $N_T$ is totally geodesic in $\bar{M}$. On the other hand, by direct calculations same as in the proof of Theorem 3.4, we deduce that $N_\perp$ is totally umbilical in $M$. Therefore, the second condition of (4.11) with (4.10) implies that $N_\perp$ is totally umbilical in $\bar{M}$. Moreover, all three conditions of (4.11) imply that $M$ is minimal submanifold of $\bar{M}$. This completes the proof of the theorem. $\blacksquare$

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