Optimal control for stochastic linear quadratic singular Takagi–Sugeno fuzzy delay system using genetic programming

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A B S T R A C T

In this paper, optimal control for stochastic linear singular Takagi–Sugeno (T–S) fuzzy delay system with quadratic performance is obtained using genetic programming (GP). To obtain the optimal control, the solution of matrix Riccati differential equation (MRDE) is computed by solving differential algebraic equation (DAE) using a novel and nontraditional GP approach. The GP solution is equivalent or very close to the exact solution of the problem. Accuracy of the GP solution to the problem is qualitatively better. The solution of this novel method is compared with the traditional Runge Kutta (RK) method. An illustrative numerical example is presented for the proposed method.

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1. Introduction

Time delay exists inevitably in active control systems which mainly results from the following:

(a) The time taken in the online data acquisition from sensors at different locations of the system.
(b) The time taken in the filtering and processing of the sensory data for the required control force calculation and the transmission of the control force to the actuator.
(c) The time taken by the actuator to produce the required control force.

Time delay systems occur in various fields such as aeronautical, astronautical, mechanical, chemical and electrical engineering. Many methods have been proposed to deal with the time delay control system [10].

A fuzzy system consists of linguistic IF–THEN rules that have fuzzy antecedent and consequent parts. It is a static nonlinear mapping from the input space to the output space. The inputs and outputs are crisp real numbers and not fuzzy sets. The fuzzification block converts the crisp inputs to fuzzy sets and then the inference mechanism uses the fuzzy rules to produce fuzzy conclusions or fuzzy aggregations and finally the defuzzification block converts these fuzzy conclusions into crisp outputs. The fuzzy system with singleton fuzzifier, product inference engine, center average defuzzifier and Gaussian membership functions is called as standard fuzzy system [30].

Two main advantages of fuzzy systems for control and modelling applications are (i) uncertain or approximate reasoning, especially difficult to express a mathematical model (ii) decision making problems with the estimated values under incomplete or uncertain information [35,36].

Genetic programming is an evolutionary algorithm that attempts to evolve solution to the given problem by using concepts taken from naturally occurring evolving process. The technique is based on the evolution of a large number of candidate solutions through genetic operations such as reproduction, crossover and mutation. It is based upon the Genetic algorithm (GA) [14], which

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exploits the process of natural selection based on a fitness measure to breed a population of trial solution that improves over time. While GA usually operates on (coded) strings of numbers but GP uses the principles and ideas from biological evolution to guide the computer to acquire desired solution. The search space is too large to attempt a brute force search, the method must be utilized to reduce the number of examined solutions. In this search, initially the population looks a bit like a cloud of randomly selected points, but that generation after generation it moves in the search space following a well defined trajectory. The generation is achieved with the help of grammatical evolution, because grammatical evolution can produce programmes in an arbitrary language, the genetic operations are faster and also because it is more convenient to symbolically differentiate mathematical expression. The code production is performed using a mapping process governed by grammar expressed in Backus Naur Form (BNF) [23]. In analogy to nature, the potential solution is an individual in some collection or population of potential solutions. The individuals who are stronger, meaning higher ranked according to fitness function, will be used to determine the next collection of potential solution. A new generation will be arisen by employing analogs of reproduction and mutation.

This means that GP has advantages over other algorithms as it can perform optimization at a structural level. This enabled Koza [17] to demonstrate the application of GP algorithm to a number of problem domains, including regression, control and classification. Research in this area has grown rapidly and encompassed a wide range of problems. GP techniques have been successfully applied in various engineering fields like signal processing [26], electrical circuit design [16], scheduling [32], process controller evolution [25] and modelling of both steady-state and dynamic processes [21].

In this paper, optimal control of stochastic linear quadratic singular T–S fuzzy delay system is obtained using genetic programming. The linear T–S fuzzy system is the most popular fuzzy model due to its further intrinsic analysis; the linear matrix inequality (LMI)-based fuzzy controller is to minimize the upper bound of the performance index; structure oriented and switching fuzzy controllers are developed for more complicated systems [27]; the optimal fuzzy control technique is used to minimize the performance index from local-concept or global-concept approaches [32,33].

Stochastic linear quadratic regulator (LQR) problems have been studied by many researchers [1,5,6,12,31]. Chen et al. [11] have shown that the stochastic LQR problem is well posed if there are solutions to the Riccati equation and then an optimal feedback control can be obtained. For LQR problems, it is natural to study an associated Riccati equation. However, the existence and uniqueness of the solution of the Riccati equation in general, seem to be very difficult problems due to the presence of the complicated nonlinear term. Zhu and Li [37] used the iterative method for solving stochastic Riccati equations for stochastic LQR problems. There are several numerical methods to solve conventional Riccati equation as a result of the nonlinear process essential error accumulations may occur. In order to minimize the error, recently the conventional Riccati equation has been analyzed using neural network approach and genetic programming approach see [2–4,29]. A variety of numerical algorithms [9] have been developed for solving the algebraic Riccati equation.

Singular systems contain a mixture of algebraic and differential equations. In that sense, the algebraic equations represent the constraints to the solution of the differential part. These systems are also known as degenerate, differential algebraic, descriptor or semi state and generalized state space systems. The complex nature of singular system causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control. The system arises naturally as a linear approximation of system models or linear system models in many applications such as electrical networks, aircraft dynamics, neutral delay systems, chemical, thermal and diffusion processes, large scale systems, robotics, biology, etc., see [7,8,20]. As the theory of optimal control of linear systems with quadratic performance criteria is well developed, the results are most complete and close to use in many practical designing problems. The theory of the quadratic cost control problem has been treated as a more interesting problem and the optimal feedback with minimum cost control has been characterized by the solution of a Riccati equation. Da Prato and Ichikawa [13] showed that the optimal feedback control and the minimum cost are characterized by the solution of the Riccati equation. Solving the MRDE is the central issue in optimal control theory.

Although parallel algorithms can compute the solutions faster than sequential algorithms, there have been no report on genetic programming solutions [29] for MRDE. This paper focuses upon the implementation of genetic programming approach for solving MRDE in order to get the optimal solution. An example is given to illustrate the advantage of GP solution.

This paper is organized as follows. In Section 2, the statement of the problem is given. In Sections 3 and 4, solution of the MRDE is presented. In Section 5, numerical example is discussed. The final conclusion Section 6 demonstrates the efficiency of the method.

2. Statement of the problem

Consider the linear dynamical singular T–S fuzzy delay system [34] that can be expressed in the form:

$$R^t: \text{if } x_0 \in T_{ij}(m_j, \sigma_j, l = 1, \ldots, r \text{ and } j = 1, \ldots, n \text{ then }}
$$

$$Fdx(t) = [A_x(t) + B_i(t - \tau)]dt + D_iw(t)\text{d}W(t), \quad x(0) = 0, \quad t \in [0, \tau],
$$

(1)

where $R^t$ denotes the ith rule of the fuzzy model, $m_j$ and $\sigma_j$ are the mean and standard deviation of the Gaussian membership function, the matrix $F$ is possibly singular, $x(t) \in R^n$ is a generalized state space vector, $u(t) \in R^m$ is a control variable and it takes value in some Euclidean space, $W(t)$ is a Brownian motion and $A \in R^{n \times n}, B \in R^{n \times m}$ and $D \in R^{m \times m}$ are known as coefficient matrices associated with $x(t)$ and $u(t)$ respectively, $x_0$ is given initial state vector and $m \leq n$.

By the following transformation [19],

$$y(t) = x(t) + \int_{-\tau}^{0} e^{-A(t + \eta)}Bu(t + \eta)d\eta,
$$

the system dynamics (1) can be rewritten into a standard form of first order differential equation with out any explicit time delay term as

$$Fdy(t) = [A_y(t) + B_i(t)]u(t)dt + D_iu(t)dW(t), \quad y(0) = 0, \quad t \in [0, \tau],
$$

(2)

where $B_i(t) = e^{-A(t)}B_iA(t)\in R^{n \times m}$.

In order to minimize both state and control signals of the feedback control system, a quadratic performance index is usually minimized:

$$J = E \left\{ \frac{1}{2}y(T)^{T}SF_{y}y(T) + \int_{0}^{T} \left[ y^{T}(t)Qy(t) + u^{T}(t)Ru(t) \right] dt \right\},
$$

where the superscript $T$ denotes the transpose operator, $S \in R^{n \times n}$ and $Q \in R^{m \times m}$ are symmetric and positive definite (or semidefinite) weighting matrices for $x(t)$, $R \in R^{m \times m}$ is a symmetric and positive definite weighting matrix for $u(t)$. It will be assumed that $\|Sf_i - A_i\| \neq 0$ for some $s$. This assumption guarantees that any input $u(t)$ will generate one and only one state trajectory $x(t)$.
If all state variables are measurable, then a linear state feedback control law
\[ u(t) = -\left( R + D^T_k(t)D_k \right)^{-1} [B_k(A_k)]^T \lambda(t) \]
can be obtained to the system described by Eq. (2), where
\[ \lambda(t) = K(t)F_k y(t), \]
\[ K(t) \in \mathbb{R}^{n \times n} \] is a symmetric matrix and the solution of MRDE and the value of \( y(t) \) can be found out using numerical algorithm.

The relative MRDE for the stochastic linear singular T–S fuzzy delay system (2) is
\[ F_k^T K(t)F_k + F_k^T K(t)A_k + A_k^T K(t)F_k + Q \]
\[- F_k^T K(t) [B_k(A_k)] \left( R + D^T_k(t)D_k \right)^{-1} [B_k(A_k)]^T K(t) F_k = 0 \] (3)
with terminal condition \( (TC) K(t_f) = F_k^T S F_k \) and \( R + D^T_k(t_f)D_k > 0 \).

After substituting the appropriate matrices in the above equation, it becomes a DAE of index one. Therefore solving MRDE is equivalent to solving the DAE of index one.

3. Solution of MRDE

Consider the DAE for (3) in each rule of the fuzzy model
\[ k_{ij}(t) = \phi_{ij}(k_i(t)), \]
\[ k_{ij}(t_f) = A_{ij} \] (4)
\[ k_{1n}(t) = \psi(k_{1n}(t)), k_{1n}(t_f) = A_{1n}. \]

3.1. Runge Kutta method

Numerical integration is one of the oldest and most fascinating topics in numerical analysis. It is the process of producing a numerical value for the integration of a function over a set. Numerical integration is usually utilized when analytic techniques fail. Even if the indefinite integral of the function is available in a closed form, it may involve some special functions, which cannot be computed easily. In such cases also, we can use numerical integration. RK algorithms have always been considered as the best tool for the numerical integration of ordinary differential equations (ODEs). The DAE can be changed into system of nonlinear differential equations by differentiating the algebraic equation one time since the DAE is of index one type. The system (3) contains \( n^2 \) first order ODEs with \( n^2 \) variables. In particular \( n = 2 \), the system will contain four equations. Since the matrix \( K(t) \) is symmetric and the system is singular, \( k_{12} = k_{21} \) and \( k_{23} \) is free (let \( k_{23} = 0 \)). Finally the system will have two equations in two variables. Hence RK method is explained for a system of two first order ODEs with two variables.

\[ k_{11}(i+1) = k_{11}(i) + \frac{1}{6} (k1 + 2k2 + 2k3 + k4) \]
\[ k_{12}(i+1) = k_{12}(i) + \frac{1}{6} (l1 + 2l2 + 2l3 + l4) \]

where
\[ k1 = h \cdot \phi_{11}(k_{11} \cdot k_{12}) \]
\[ l1 = h \cdot \phi_{12}(k_{11} \cdot k_{12}) \]
\[ k2 = h \cdot \phi_{11} \left( k_{11} + \frac{k1}{2}, k_{12} + \frac{l1}{2} \right) \]
\[ l2 = h \cdot \phi_{12} \left( k_{11} + \frac{k1}{2}, k_{12} + \frac{l1}{2} \right) \]
\[ k3 = h \cdot \phi_{11} \left( k_{11} + \frac{k2}{2}, k_{12} + \frac{l2}{2} \right) \]
\[ l3 = h \cdot \phi_{12} \left( k_{11} + \frac{k2}{2}, k_{12} + \frac{l2}{2} \right) \]
\[ k4 = h \cdot \phi_{11}(k_{11} + k3, k_{12} + l3) \]
\[ l4 = h \cdot \phi_{12}(k_{11} + k3, k_{12} + l3) \]

In the similar way, the original system (4) can be solved for \( n^2 \) first order ODE’s.

4. Genetic programming method

In this approach, GP is used to obtain a set of expressions. If the required number expressions satisfy the fitness function, it will be the optimal solution of MRDE. The scheme of computing optimal solution is given in Fig. 1.

4.1. Initialization of the population

The first step is to initialize the population. An initial population of the desired size is generated randomly. The length of each chromosome is to be set according to the nature of the problem. Each program or individual in the population is generally represented as a Parse tree composed of function and data/terminals appropriate to the problem domain.

4.2. Grammatical evolution

Grammatical evolution is an evolutionary algorithm that can produce code in any programming language. The algorithm starts from the start symbol of the grammar and gradually creates the program string, by replacing non-terminal symbol with right hand of the selected production rule, first read an element from the chromosome (with value \( V \)) and compute Rule = \( V \) mod NR, where NR is...
the number of rules for the specific non-terminal symbol which is shown in Table 1.

The symbol S in the grammar denotes the start symbol of the grammar. For example, suppose we have the chromosome \( x = \{ 1, 2, 10, 4, 2, 11, 16, 30, 5 \} \). The method of producing a valid expression is shown in Table 2. The rule \( V \) is applied in each row of Table 2. In first line \( V = 7, NR = 7 \) the number of elements in Table 2. The next line \( \exp < \op < \exp > \) is obtained from Table 1 for the result zero of the first line. Similarly the remaining rows of Table 2 can be found. The function for the chromosome is \( \exp (x) + \log (\exp (y)) \), (for details see [15,18,24,28]).

### 4.3. Fitness function

The aim of the fitness function is to provide a basis for competition among available solutions and to obtain the optimal solution. Hence, the fitness function for MRDE is defined as

\[
E_r = (k_1^n - \phi(k_{ij}))^2 + \sum_{i,j=1}^{n-1} (k_{ij} - \phi_i(k_{ij}))^2
\]  

(5)

Table 2

<table>
<thead>
<tr>
<th>String</th>
<th>Chromosome</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;exp&gt;&lt;op&gt;&lt;exp&gt;</td>
<td>7, 2, 10, 4, 2, 11, 16, 30, 5</td>
<td>7 mod 7 = 0</td>
</tr>
<tr>
<td>&lt;exp&gt;&lt;op&gt;&lt;exp&gt;</td>
<td>10, 4, 2, 11, 16, 30, 5</td>
<td>10 mod 4 = 2</td>
</tr>
<tr>
<td>*</td>
<td>4, 4, 2, 11, 16, 30, 5</td>
<td>4 mod 4 = 0</td>
</tr>
<tr>
<td>*</td>
<td>2, 11, 16, 30, 5</td>
<td>2 mod 7 = 2</td>
</tr>
<tr>
<td>+</td>
<td>11, 16, 30, 5</td>
<td>11 mod 4 = 3</td>
</tr>
<tr>
<td>+</td>
<td>16, 30, 5</td>
<td>16 mod 7 = 2</td>
</tr>
<tr>
<td>*</td>
<td>30, 5</td>
<td>30 mod 4 = 2</td>
</tr>
<tr>
<td>*</td>
<td>5</td>
<td>5 mod 7 = 5</td>
</tr>
</tbody>
</table>

### 4.4. Genetic operators

The genetic operators such as reproduction, crossover and mutation are explained below:

**Reproduction**: In reproduction process, best chromosome in a population is probabilistically assigned a large number of copies according to their fitness value. It is important to note that no new strings are formed in the reproduction phase. Koza [17] allowed 10 percentage of the population to reproduce.

**Crossover**: The crossover is applied every generation in order to create new chromosome from the old ones, that will replace the worst individuals in the population. There are many types of crossover operator in which we use single point crossover technique. In that operation, for each couple of new chromosomes two parents are selected, we cut these parent - chromosomes at a randomly chosen point and exchange the right-hand-side sub-chromosomes, as shown in Fig. 2. A random number is generated for each chromosome. If the random number is less than the crossover probability, then the chromosome is chosen for crossover with previously chosen chromosome.

**Mutation**: In the mutation process, for every element in a chromosome a random number in the range [0, 1] is chosen. If the number is less than or equal to the mutation rate, the corresponding element is changed randomly. Otherwise it remains intact.

### 4.5. Termination control

In each generation, a set of expressions are generated by the chromosomes. If an expression minimizes the fitness function \( E_r \) to zero or very close to zero and satisfies the initial condition, the process may be stopped. Otherwise, the GP approach must be continued.

### 4.6. Genetic programming algorithm

The algorithm has the following steps

1. Initialize random population.
2. Create valid function using grammar.
3. Evaluate fitness value of the chromosome.
4. If Fitness tends to zero, stop the procedure. Otherwise proceed to next step.
5. Generate new population using Genetic operations, Go to step 2.

![Fig. 2. One point crossover.](image-url)
5. Numerical example

Consider the optimal control problem:

\[
J = E \left\{ \frac{1}{2} y^T(t_f) \delta S F y(t_f) + \frac{1}{2} \int_0^{t_f} [y^T(t)Qy(t) + u^T(t)Ru(t)]dt \right\}
\]

subject to the stochastic linear singular T–S fuzzy system \( R^i \): If \( y_1 \) is \( T_1(m_{1i}, \sigma_{1i}), i = 1, 2, \) then

\[
F_i \delta y(t) = [A_j y(t) + B_j(A_j^i)u(t)]dt + D_i u(t)dW(t), \quad y(0) = y_0
\]

where

\[
S = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -2 \\ 0 & 2 \end{bmatrix},
\]

\[
B_j(A_j^i) = \begin{bmatrix} e^\tau - e^{2\tau} \\ -e^\tau + 2e^{2\tau} \end{bmatrix}, \quad R = 1, \quad Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

The numerical implementation could be adapted by taking \( t_f = 2 \) and \( \tau \to 0 \) for solving the related MRDE of the above linear singular fuzzy system. The appropriate matrices are substituted in Eq. (3), the MRDE is transformed into DAE in \( k_{11} \) and \( k_{12} \). In this problem, the value of \( k_{22} \) of the symmetric matrix \( K_0(t) \) is free and let \( k_{22} = 0 \). Then the optimal control of the system can be found out by the solution of MRDE.

5.1. Solution obtained using genetic programming

To solve the MRDE, each chromosome is split uniformly in 23 parts and Population size is 100. In the computation process, the parameters are taken as 0.01 for replication rate, 0.9 for the crossover probability and 0.05 for the mutation rate.

After 700 generations, the following expressions satisfy the fitness function and the value of the fitness function tends to 0. The expressions also satisfy the terminal conditions. Hence the solution of the DAE/MRDE is obtained.

Expressions: \( k_{11} = 2e^{3(t-2)}, k_{12} = 2e^{3(t-2)} + 1 \)

The numerical solutions of MRDE are calculated and displayed in Table 3 using the GP and RK methods. GP solution curves are shown in Figs. 3 and 5. From Table 3 and Figs. 3 and 5, GP solution is equal to the exact solution of MRDE. The parse trees for the solutions are given in Figs. 4 and 6.

Similarly the solution of the above system with the matrix \( A_2 \) can be found out using genetic programming.
6. Conclusion

The optimal control for the stochastic linear singular T–S fuzzy delay system has been found by GP approach. To obtain the optimal control, the solution of MRDE is computed by solving differential algebraic equation (DAE) using a novel and nontraditional GP approach. The GP solution is equal to the exact solution of the problem. Accuracy of the GP solution to the problem is qualitatively better when it is compared with RK solution. A numerical example is given to illustrate the derived results.

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Fig. 6. Tree and expression for $k_{2,3}$. 