Stochastic dynamics of charge fluctuations in dusty plasma: A non-Markovian approach

H. Asgari, a) S. V. Muniandy, and C. S. Wong

Plasma Research Laboratory, Department of Physics, University of Malaya, Kuala Lumpur 50603, Malaysia

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Dust particles in typical laboratory plasma become charged largely by collecting electrons and/or ions. Most of the theoretical studies in dusty plasma assume that the grain charge remains constant even though it fluctuates due to the discrete nature of the charge. The rates of ions and electrons absorption depend on the grain charge, hence its temporal evolution. Stochastic charging model based on the standard Langevin equation assumes that the underlying process is Markovian. In this work, the memory effect in dust charging dynamics is incorporated using fractional calculus formalism. The resulting fractional Langevin equation is solved to obtain the amplitude and correlation function for the dust charge fluctuation. It is shown that the effects of ion-neutral collisions can be interpreted in phenomenological sense through the nonlocal fractional order derivative. © 2011 American Institute of Physics. [doi:10.1063/1.3626552]

I. INTRODUCTION

When dust grains are exposed to plasma, they become charged by collecting electrons and ions and sometimes by emitting processes such as thermionic, secondary electron emission, and photoelectric emission. Dust charge and its fluctuation can strongly affect properties of dusty plasma systems. 1–9 For example, dust charge fluctuation perturbs both electrostatic and Lorentz force such that the dust motion is reciprocally affected. 1 In the case of dust crystals, charge fluctuation leads to intergrain potential fluctuation that causes temporal variation in the coupling strength thereby inhibiting crystallization. 1 Another example is the experimentally verified grain heating influenced by charge fluctuation in which unexpectedly high temperature is observed (up to ≈ 50 eV). 10

One cause of charge fluctuation is the spatial and temporal variation in the surrounding plasma. 2,3 Another reason which is of our main concern in this paper is the charge fluctuation due to the discrete nature of the charges. The time interval between the successive absorption of plasma particles and also the random sequence of the events in which electrons and ions reach the grain surface contribute to charge fluctuation even in steady-state uniform plasma. Dust charge fluctuation due to the discrete nature of charge has been investigated in several studies. 1,4–9 Matsoukas and Russell 4 presented a Fokker-Planck description of the particle charging in ionized gas and charge fluctuation arising from the statistical nature of the process and showed that the time scale of fluctuation is inversely proportional to the particle size and ion concentration. They also developed an analytical model of stochastic charge fluctuation of dust particles in order to quantify the statistical properties of charge fluctuation such as charge distribution and the amplitude of fluctuation for both Maxwellian and non-Maxwellian electrons. 5–6 Choi and Kushner 7 carried out particle-in-cell (PIC) simulation of the transport of ions and electrons in the vicinity of dust particles. The simulation was used to describe the electrical charge on the dust particle and by plotting the total charge on the particle as a function of time revealed the fluctuation. 7 Based on the Poisson statistic, Morfill et al. 8 assumed that the number of charges on a dust grain changes randomly as \( \Delta Z = \langle |Z| \rangle^{1/2} \) where \( \langle Z \rangle \) is the equilibrium dust grain charge in units of electron charges. Cui and Goree 1 developed a physical model in which the rate of absorbing ions and electrons follows probabilities distribution which depends on the dust potential. By running a numerical simulation, they derived that the root mean square distribution which depends on the dust potential. An analytical model represented by Khrapak et al. 9 showed that for all charging mechanisms, the amplitude of fluctuation varies as \( \Delta Z = \alpha \langle |Z| \rangle^{1/2} \) where coefficient \( \alpha \) as well as the characteristic fluctuation frequency can be obtained once the specific form of the charging currents is known.

Time interval between the absorption of plasma particles and the sequence in which electrons and ions arrive at the grain surface can be considered as random variables. Neither of these processes is mutually independent. The probability of absorbing plasma particles depends on the grain potential. 1 As the grain potential becomes more negative, more electrons will be repelled and more ions will be attracted to the grain. Hence, the probability of attracting ion increases with potential while the probability of attracting electrons decreases. The rate of absorbing plasma species depends on the grain’s charge and potential and, therefore, on the history of events. It means the evolution of the system at time \( t \) depends on its past or the dynamical process is said to be non-Markovian. On the contrary, Poisson statistics describes random events where the probability for one event does not depend on the history of previous events. Such a process is not suitable for describing temporally correlated systems or processes with (short or long) memory. In this work, the memory effect in dust charging dynamics is incorporated.

a)Electronic mail: hosseinasgari@perdana.um.edu.my.
through the use of fractional calculus. The standard Langevin equation with Markovian solution is generalized to fractional Langevin equation using the β-th order fractional derivative and its shifted definition. This results in a non-Markovian process known as fractional Ornstein-Uhlenbeck (OU) process which can be used to describe the memory effect in the charging process. The consistency of the model is verified by its reduction to ordinary Langevin equation with Markovian solution (Ornstein-Uhlenbeck process or oscillator process) when β = 1. Finally, we briefly demonstrate one possible application of the fractional Ornstein-Uhlenbeck process in providing a better description of the results obtained from experiment on determining dust charge.

II. THEORETICAL MODEL AND RESULTS

A dust grain immersed in plasma will gradually get charged by collecting electron and ion currents according to charge dynamic equation given by

\[
\frac{dZ}{dt} = \sum I_z.
\]

The temporal evolution of charge number \( Z(t) \) can be found by integrating Eq. (1) using suitable expressions for the currents \( I_z \). The equilibrium charge \( \langle Z \rangle \) is obtained by putting the right hand side of Eq. (1) equal to zero. If one assumes that the charge on the dust fluctuates, the charge number is then a time-dependent variable

\[
\delta Z(t) = \langle Z \rangle + \delta Z(t),
\]

where \( \delta Z(t) \) indicates small fluctuation from the equilibrium charge \( \langle Z \rangle \) and it is often assumed that \( |\langle Z \rangle| \gg |\delta Z(t)| \). By keeping only terms up to the first order, it is possible to write Eq. (1) in the following form:

\[
\frac{d\delta Z}{dt} = \left[ \frac{\partial I}{\partial Z} \right]_{x=\langle x \rangle} \delta Z = -\gamma \delta Z,
\]

where \( \gamma \delta Z \) with \( (\gamma > 0) \) acts as a restoring force that tends to bring the charge back to the equilibrium value conditioned by \( I = 0 \).

In order to extend the charging model to include the effects of discrete charge fluctuation, we rewrite the charge dynamics equation [Eq. (3)] by adding a noise term responsible for the fluctuation in both directions around the equilibrium

\[
\frac{d\delta Z}{dt} + \gamma \delta Z = F(t),
\]

where \( F(t) \) is the stochastic noise term given by

\[
F(t) = \sum_i \delta(t - t_i)(\pm 1),
\]

referring to the absorption of electron/ion or to the emission of an electron at random time \( t_i \). Equation (5) can be solved in analogy with the Langevin equation of motion of a free Brownian particle in one dimension if one treats \( \delta Z \) as a particle velocity and \( F(t) \) as a Gaussian white noise which satisfies

\[
\langle F(t) \rangle = 0, \quad \langle F(t_1)F(t_2) \rangle = \frac{1}{t_0} \delta(t_1 - t_2).
\]

Here, \( 1/t_0 \) characterizes the rate of particle absorption and emission.

Recall that the sequences in which electrons and ions reach the grain surface or the order of electron emission by the grain can be treated as random processes. The time intervals between absorption of plasma particles or emission also vary in a random manner, but these processes are not mutually independent. The rates of ions and electrons absorption obey probabilities that depend on the grain potential and, therefore, on the history of absorption. In order to include the effects of memory, we generalize the standard Langevin formalism [Eq. (4)] to two different forms of "fractional Langevin equation." First let us consider the generalization in which we replace the first order derivative with respect to time in Eq. (4) with \( z \)-th order fractional derivative

\[
aD_z^\alpha \delta Z + \gamma \delta Z = F(t),
\]

where \( aD_z^\alpha \) is the Riemann-Liouville fractional derivative with \( 0 \leq \alpha < 1 \), defined as

\[
aD_z^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \frac{f(t')}{(t' - t)^{\alpha}} dt'.
\]

On the other hand, the Riemann-Liouville fractional integral is written as

\[
J_z^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t')}{(t' - t)^{1-\alpha}} dt'.
\]

Equation (7) can be solved\(^a\) by applying the above fractional integral operator to both sides of this equation to obtain

\[
\delta Z(t) - \delta Z(0) = J_z^\alpha (\gamma \delta Z(t) + F(t)).
\]

Assuming the initial deviation of the grain charge equal to zero i.e., \( \delta Z(0) = 0 \), Eq. (10) can be solved by standard techniques and its solution is

\[
\delta Z(t) = \int_0^t \frac{F(t')}{(t' - t)^{\alpha - 1}} E_{\alpha,\alpha} \left[ -\gamma (t' - t)^{\alpha - 1} \right] dt',
\]

where \( E_{\alpha,\alpha}(x) = \sum_{k=0}^{\infty} x^k / \Gamma(\alpha + \beta k) \) is the Mittag-Leffler function. Provided that \( F(t) \) satisfies Eq. (6), the temporal correlation function of grain charge fluctuation has the form

\[
\langle \delta Z(t_1) \delta Z(t_2) \rangle = \frac{1}{t_0} \int_0^{\min(t_1, t_2)} \frac{E_{\alpha,\alpha}[-\gamma (t_1 - t')^{\alpha - 1}] E_{\alpha,\alpha}[-\gamma (t_2 - t')^{\alpha - 1}]}{(t_1 - t')^{\alpha - 2}(t_2 - t')^{\alpha - 2}} dt',
\]

assuming that \( t_1 < t_2 \). By using the general form of the Mittag-Leffler function, the integral in Eq. (12) can be written as
\[ \langle \delta Z(t_1) \delta Z(t_2) \rangle = \frac{1}{t_0} \int_0^1 \left( \sum_{k=0}^{\infty} \frac{(-\gamma)^k (t_1 - t')^{(k+1)-1}}{\Gamma(z' + 1)} \right) \sum_{k=0}^{\infty} \frac{(-\gamma)^k (t_2 - t')^{(k+1)-1}}{\Gamma(z(k' + 1))} dt'. \]

(13)

In order to solve the integral in Eq. (13), we assume that \( t'/t_2 < 1 \) since \( t_1 < t_2 \). We then obtain

\[ \langle t_2 - t' \rangle x^{(k+1)-1} = t_2 x^{(k+1)-1} \left( 1 - \frac{t'}{t_2} \right)^{x^{(k+1)-1}} = t_2 x^{(k+1)-1} \left[ 1 - (\gamma(k' + 1) - 1) \frac{t'}{t_2} + \cdots \right]. \]

(14)

By substituting Eq. (14) in Eq. (13), it is possible to solve the integral for different order of expansion using

\[ \int_0^1 x^a (a - x)^n dx = \frac{a^{1+m+n}}{\Gamma(1+m)\Gamma(2+m+n)}. \]

(15)

Finally, one obtains the correlation function for charge fluctuation as

\[ \langle \delta Z(t_1) \delta Z(t_2) \rangle = \sum_{n=0}^{\infty} (-1)^n \frac{a^{1+m+n}}{t_0} E_{x+n+1,2} (-\gamma t_1^2) E_{x-n,2} (-\gamma t_2^2). \]

(16)

One can see that the correlation function of charge fluctuation depends explicitly on both times \( t_1 \) and \( t_2 \) and not only on the difference \( t_1 - t_2 \), indicating what is known as second-order non-stationary behavior. Due to this complication, the direct introduction of the Riemann-Liouville fractional derivative into the Langevin formalism may not be helpful for understanding non-Markovian effect in the charging dynamics. Nevertheless, this framework may be used to describe a broader class of non-equilibrium phenomena.

Next, we consider another possible generalization of Langevin formalism for dust charging that satisfies second-order stationary property. We start with the standard Langevin equation written in the following form

\[ \left( \frac{d}{dt} + a \right) X_{OU}(t) = w(t), \]

(17)

where \( a \) is a positive constant related to characteristic time of the damped oscillator process and \( w(t) \) is a white noise with zero mean and delta function as a correlation function, i.e., \( \langle w(t)w(t') \rangle = \delta(t - t') \). The solution of Eq. (17) is called the OU process or the oscillator process with Markovian characteristics. The similarity between OU process and dust charge fluctuation described by Eq. (4) is quite obvious. The stationary solution of Eq. (17) can be written as

\[ X_{OU}(t) = \int_{-\infty}^{t} e^{-a(t-t')} w(t') dt'. \]

(18)

and the correlation function is given as

\[ C(\tau) = \langle X_{OU}(t)X_{OU}(t + \tau) \rangle = \exp(-a|\tau|)/(2a). \]

It is easy to show that the OU process satisfies the time-ordered semi-group property for it to be a Markovian process and indeed it also exhibits short range correlation as defined by the integrability of the correlation function. Now, we introduce the generalized version of the OU process using the shifted fractional derivation notation as follows:

\[ \left( \frac{d}{dt} + a \right)^{\beta} X_{\beta}(t) = w(t), \]

(19)

with fractional index \( \beta > 0 \). To incorporate fractional charging dynamics, Eq. (4) take the form of

\[ \left( \frac{d}{dt} + \gamma \right)^{\beta} \delta Z_{\beta} = F(t). \]

(20)

Equation (20) can be considered as a non-standard form of “fractional Langevin equation” which can be solved by Green’s function technique. Considering \( \delta Z_{\beta}(t) \) as the impulse response function of the system, we write

\[ \left( \frac{d}{dt} + \gamma \right)^{\beta} \delta Z_{\beta} = \delta(t). \]

(21)

By applying Fourier transform on both sides of Eq. (21) and considering the property of Fourier transform of fractional derivatives, \( \gamma \)

\[ S_{\delta Z_{\beta}}(t) = Z_{\beta}(\omega)\delta Z_{\beta}(\omega) = \frac{1}{(\gamma^2 + \omega^2)^{\gamma}}, \]

(23)

where \( |Z_{\beta}(\omega)|^2 \) is the complex conjugate of \( Z_{\beta}(\omega) \). The solution of Eq. (20) is written as

\[ \delta Z_{\beta} = c(a, \beta) \int_{-\infty}^{t} z_{\beta}(t - t') F(t') dt'. \]

(24)

with \( c(a, \beta) \) as an arbitrary constant which can be determined by choosing \( \beta = 1 \).

Using the Wiener-Khinchine theorem \( \gamma \) that links the power spectral density \( S_{\delta Z_{\beta}}(\omega) \) of a stationary process to its correlation function \( C_{\beta}(\tau) \), one obtains

\[ C_{\beta}(t_2 - t_1) = \langle \delta Z(t_1) \delta Z(t_2) \rangle = \frac{\gamma^{\gamma-1}}{2^\gamma \pi^{\gamma/2}} |\tau(t_2 - t_1)|^{\gamma-1} K_{\nu}(|\tau(t_2 - t_1)|) \]

(25)

with \( \nu = \beta - 1/2 \) and \( K_{\nu} \) is the modified Bessel function of the second kind. The first advantage of using the fOU process to describe dust charge dynamics is that the correlation
fractional index function of the second kind $K_{kv}$.

The two main effects for charge reduction are that of “closely packed” grains, leading to a drastic charge suppression (up to 5 times), and the effect of “closely packed” grains is not important. The quasi-neutrality condition is weakly affected by the dust injection. Experimental determination of parameters of the dust charge suppression via ion-neutral collisions, we consider a fractional charging model was shown to be useful for describing the memory of the stochastic charging process through

$$\phi = \int_0^\infty C(t)dt$$

(26)

As noted before, a process is said to exhibit short-memory if $\phi$ is finite and a long-memory process if $\phi$ diverges. The fOU process can be shown to be non-Markovian process by verifying that $C_p(t_3-t_1) = C_p(t_3-t_2) \times C_p(t_2-t_1)$ for $t_1 < t_2 < t_3$ and also exhibits short memory with rate of decay slower compared to the standard exponential form of OU process. As one may expect in the limit of $\nu \to 1/2$ or $\beta \to 1$, by setting $t_1 = t_2$ in Eq. (25), one can calculate the rms amplitude of fluctuation, namely $\Delta Z \equiv \sqrt{\langle \delta Z^2 \rangle} = (2\nu)^{-1/2}$. A similar expression for amplitude of fluctuation was obtained in Ref. 9.

Next, we show the applicability of fractional charging dynamical model by demonstrating how the fOU can provide a better match between theory and experimental measurements of the dust charge. Experimental determination of particle charge in a bulk dc discharge shows some discrepancies between experiments and orbital motion limited (OML) theory, especially at high pressure. It was claimed that the two main effects for charge reduction are that of “closely packed” grains and ion-neutral collisions. In those experiments with $\Delta/\lambda_D > 1$, (where $\Delta$ is the intergrain distance) the quasi-neutrality condition is weakly affected by the dust and the effect of “closely packed” grains is not important and, therefore, the drastic charge suppression (up to 5 times) is attributed to the effect of ion-neutral collisions. Based on fOU process, we propose a phenomenological interpretation of memory effect of ion-neutral collision as nonlocal effect of fractional derivative operator parameterized by the fractional index $\beta$.

By using the series expansion of the modified Bessel function of the second kind $K_{\nu}$, in form of

$$k_{\nu}(x) = 2^{\nu-1}(v-1)!x^{-\nu} + \cdots$$

the following relation is derived from Eq. (25) for the rms amplitude of fluctuations:

$$\Delta Z = \sqrt{\langle \delta Z^2 \rangle} = \frac{\gamma^{-2v}(v-1)!}{2\nu\sqrt{\pi}Gamma(v+1/2)},$$

(27)

with $\nu = \beta - 1/2$. It is well-known that $\Delta Z \approx \sqrt{\langle |Z| \rangle}$, which means the bigger rms amplitude of the fluctuation implies bigger average charge residing on the dust. To illustrate the influence of slight variation in the fractional index on the charge suppression via ion-neutral collisions, we consider a dusty plasma system with $\gamma(s^{-1}) \sim 10^2$, and by solving the following equation:

$$\frac{\Delta Z|_{\beta = 1+\varepsilon}}{\Delta Z|_{\beta = 1}} = \frac{\sqrt{\langle |Z| \rangle|_{\beta = 1+\varepsilon}}}{\sqrt{\langle |Z| \rangle|_{\beta = 1}}} = \frac{1}{2}. $$

(28)

we obtain $\varepsilon = 0.035$. It means a slight change in fractional index from $\beta = 1$ to $\beta = 1.035$ will result in the suppression of charge up to 4 times as one may expect from experimental results. Thus, the incorporation of non-local effect via the fractional derivatives may be useful for describing memory or non-Markovian effect.

### III. CONCLUSION

The memory effect in dust charging dynamics has been incorporated in the dust charging dynamical equation through the use of fractional derivative. Two different generalizations of the standard Langevin formalisms were considered. A direct generalization to the standard fractional Langevin equation would lead to a non-stationary non-Markovian process. In this work, we have opted for the phenomenological charging model based on the shifted fractional derivative that resulted in a stationary non-Markovian process with known as fractional Ornstein-Uhlenbeck process with Gaussian probability distribution and short range correlation. The correlation function of the dust charge fluctuation of this process is parameterized by fractional index $\beta$ and it decays slower than the exponentially correlated Ornstein-Uhlenbeck process. The fractional charging model was shown to be useful for describing the dust charge suppression due to non-local effect of ion-neutral collisions.

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