Complex Analysis

Coefficient estimates for a class of meromorphic bi-univalent functions

Estimation de coefficients pour une classe de fonctions méromorphes bi-univalentes

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1. Introduction

Let $\Sigma$ be the family of meromorphic functions $g$ of the form:

$$ g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}, \quad (1.1) $$

that are univalent in $\Delta := \{z: 1 < |z| < \infty\}$. The coefficients of $h = g^{-1}$, the inverse map of $g$, are given by the Faber polynomial:

$$ h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \sum_{n \geq 1} \frac{1}{n} K_{n+1} \frac{1}{w^n}. $$

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where \( w \in \Delta \),
\[
K_{n+1}^n = nb_n^{-1}b_1 + n(n-1)b_n^{-2}b_2 + \frac{1}{2}n(n-1)(n-2)b_n^{-3}(b_3 + b_4^2)
+ \frac{n(n-1)(n-2)(n-3)}{3!}b_n^{-4}(b_4 + 3b_1b_2) + \sum_{j=5}^{\infty} b_n^{-j}V_j
\]
and \( V_j \) with \( 5 \leq j \leq n \) is a homogeneous polynomial of degree \( j \) in the variables \( b_1, b_2, \ldots, b_n \). (See [1,2] or [11].)

For \( 0 \leq \alpha < 1 \) and \( \lambda \geq 1 \), let \( B \Sigma(\alpha; \lambda) \) be the class of meromorphic bi-univalent functions \( g \in \Sigma \) so that:
\[
\text{Re} \left( 1 - \lambda \right) \left( \frac{g(z)}{z} + \lambda g'(z) \right) > \alpha, \quad z \in \Delta
\]
and
\[
\text{Re} \left( 1 - \lambda \right) \left( \frac{h(w)}{w} + \lambda h'(w) \right) > \alpha, \quad w \in \Delta.
\]

In 1923, Lowner [9] proved that the inverse of the Koebe function \( k(z) = z/(1 - z)^2 \) provides the best upper bounds for the coefficients of the inverses of analytic univalent functions. Although the estimates for the coefficients of the inverses of analytic univalent functions have been obtained in a surprisingly straightforward way (e.g., see [7, p. 104]), the case turns out to be a challenge when the bi-univalency condition is imposed on these functions. Finding bounds for the coefficients of the inverses of analytic univalent functions has been obtained in a surprisingly straightforward way (e.g., see [7, p. 104]), the case turns out to be a challenge when the bi-univalency condition is imposed on these functions. Finding bounds for the coefficients of classes of bi-univalent functions dates back to 1967 (see Lewin [8]). The interest on the bounds for the coefficients of classes of bi-univalent functions picked up by the publications [4,10,6,3], where the estimates for the first two coefficients of certain classes of bi-univalent functions were provided. Not much is known about the higher coefficients of bi-univalent functions as Ali et al. [3] also declared finding the bounds for \( |a_n|; n \geq 4 \) an open problem. In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients \( |a_n| \) of meromorphic bi-univalent functions in \( B \Sigma(\alpha; \lambda) \) as well as providing estimates for the early coefficients of these functions. As a result, we are able to prove:

**Theorem 1.1.** Let \( g \) be given by (1.1). For \( 0 \leq \alpha < 1 \) and \( \lambda \geq 1 \) if \( g \in B \Sigma(\alpha; \lambda) \) and \( b_k = 0; 0 \leq k \leq n - 1 \), then:
\[
|b_n| \leq \frac{2(1-\alpha)}{(n+1)\lambda - 1}; \quad n \geq 1.
\]

**Proof.** For meromorphic functions \( g \) of the form (1.1) we have:
\[
(1 - \lambda) \frac{g(z)}{z} + \lambda g'(z) = 1 + \sum_{n=0}^{\infty} (1 - \lambda(n+1)) \frac{b_n}{z^{n+1}},
\]
and for its inverse map, \( h = g^{-1} \), we have:
\[
(1 - \lambda) \frac{h(w)}{w} + \lambda h'(w) = 1 + \sum_{n=0}^{\infty} (1 - \lambda(n+1)) \frac{B_n}{w^{n+1}}
= 1 - (1 - \lambda) \frac{b_0}{w} - \sum_{n=1}^{\infty} (1 - \lambda(n+1)) \frac{1}{n} K_{n+1}^n(b_0, b_1, \ldots, b_n) \frac{1}{w^{n+1}}.
\]

On the other hand, since \( g \in B \Sigma(\alpha; \lambda) \), by definition, there exist two positive real-part functions \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^{-n} \) and \( q(w) = 1 + \sum_{n=1}^{\infty} d_n w^{-n} \), where \( \text{Rep}(z) > 0 \) and \( \text{Req}(w) > 0 \) in \( \Delta \) so that:
\[
(1 - \lambda) \frac{g(z)}{z} + \lambda g'(z) = 1 + (1 - \alpha) \sum_{n=1}^{\infty} K_n^1(c_1, c_2, \ldots, c_n) \frac{1}{z^n},
\]
and
\[
(1 - \lambda) \frac{h(w)}{w} + \lambda h'(w) = 1 + (1 - \alpha) \sum_{n=1}^{\infty} K_n^1(d_1, d_2, \ldots, d_n) \frac{1}{w^n}.
\]
Note that, according to the Caratheodory Lemma (see Duren [5, p. 41]), \( |c_n| \leq 2 \) and \( |d_n| \leq 2 \) for \( n = 1, 2, 3, \ldots \). Comparing the corresponding coefficients of (1.2) and (1.4) yields:
\[
(1 - \lambda(n+1))b_n = (1 - \alpha) K_{n+1}^1(c_1, c_2, \ldots, c_{n+1})
\]
Theorem 1.2. Let \( g \) be given by (1.1). For \( 0 \leq \alpha < 1 \) and \( \lambda \geq 1 \) if \( g \in B \Sigma(\alpha; \lambda) \), then

i) \( |b_0| \leq \frac{2(1-\alpha)}{\lambda-1} \),

ii) \( |b_1| \leq \frac{2(1-\alpha)}{2\lambda-1} \),

iii) \( |b_2| \leq \frac{2(1-\alpha)}{3\lambda-1} \),

iv) \( |b_2+b_0b_1| \leq \frac{2(1-\alpha)}{3\lambda-1} \).

Proof. Comparing Eqs. (1.2) and (1.4) for \( n = 0, 1, 2 \), we obtain:

\[
\begin{align*}
(1-\lambda)b_0 &= (1-\alpha)c_1, \\
(1-2\lambda)b_1 &= (1-\alpha)c_2, \\
(1-3\lambda)b_2 &= (1-\alpha)c_3.
\end{align*}
\]

and

\( (1-3\lambda)b_2 = (1-\alpha)c_3 \).

On the other hand, from (1.3) and (1.5), for \( n = 2 \), we obtain:

\( -(1-3\lambda)(b_0b_1+b_2) = (1-\alpha)d_3 \).

Solving Eqs. (1.6), (1.7), (1.8) and (1.9) for \( b_0, b_1, b_2 \) and \( (b_2 + b_0b_1) \), respectively, taking their absolute values and then applying the Caratheodory Lemma, we obtain:

\[
\begin{align*}
|b_0| &\leq \frac{(1-\alpha)|c_1|}{|1-\lambda|} \leq \frac{2(1-\alpha)}{\lambda-1}, \\
|b_1| &\leq \frac{(1-\alpha)|c_2|}{|1-2\lambda|} \leq \frac{2(1-\alpha)}{2\lambda-1}, \\
|b_2| &\leq \frac{(1-\alpha)|c_3|}{|1-3\lambda|} \leq \frac{2(1-\alpha)}{3\lambda-1},
\end{align*}
\]

and

\[
|b_2 + b_0b_1| \leq \frac{(1-\alpha)|d_3|}{|1-3\lambda|} \leq \frac{2(1-\alpha)}{3\lambda-1}.
\]

References