CLASSES OF FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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Abstract

In this paper, new classes of functions defined using Dziok-Srivastava operator are introduced. Some inclusion theorems are determined for functions in these classes. Furthermore, certain integral operators are considered to be in the classes.

1. Introduction

Let $S$ denotes the class of all analytic functions $f$ in the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0, f'(0) = 1$. An analytic function $f$ is subordinate to an analytic function $g$, written $f(z) \prec g(z)$ ($z \in D$), if there exists an analytic function $w$ in $D$ such that $w(0) = 0$ and
\[ |w(z)| < 1 \text{ for } |z| < 1 \text{ and } f(z) = g(w(z)). \] In particular, if \( g \) is univalent in \( D \), then \( f(z) \prec g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(D) \subset g(D) \).

For complex parameters \( \alpha_i (i = 1, 2, \ldots, l) \) and \( \beta_j \in \mathbb{C}\{0, -1, -2, \ldots\} \) \((j = 1, 2, \ldots, m)\), the generalized hypergeometric function \( \genfrac(){0pt}{1}{l}{m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \) is given by

\[
\genfrac(){0pt}{1}{l}{m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}
\]

\((l \leq m + 1; l, m \in N_0 := N \cup \{0\}; z \in D)\),

where \( (\lambda)_n \) is the Pochhammer symbol defined, in terms of the gamma function \( \Gamma \), as follows:

\[
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \lambda \neq 0 \\ \lambda(\lambda + 1)(\lambda + 2)\cdots(\lambda + n - 1), & n = 1, 2, 3, \ldots \end{cases}
\]

Let \( \varphi(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( \psi(z) = \sum_{n=0}^{\infty} b_n z^n \) be analytic functions. The convolution of these functions is defined as

\[
\varphi(z) \ast \psi(z) = \sum_{n=0}^{\infty} a_n b_n z^n = \psi(z) \ast \varphi(z).
\]

Dziok and Srivastava in [5] introduced the unified linear operator

\[
H^{l,m}[\alpha_1]f(z) = z \genfrac(){0pt}{1}{l}{m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \ast f(z)
\]

which includes well known operators such as the Hohlov operator [7], Carlson-Shaffer operator [3], Ruscheweyh derivative operator [17], the generalized Bernardi-Libera-Livington integral operator [2, 12, 14] and the Srivastava-Owa fractional derivative operator [23]. It can easily be verified that

\[
\alpha_1 H^{l,m}[\alpha_1 + 1]f(z) = z[H^{l,m}[\alpha_1]f(z)]' + (\alpha_1 - 1)H^{l,m}[\alpha_1]f(z).
\]
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In particular, Bernardi’s integral operator given below:

\[ F(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) \, dt, \quad c \in \mathbb{C}, \text{ Re } c \geq 0, \]

\[ = z + \sum_{n=2}^{\infty} \frac{c + 1}{n + c} a_n z^n \]

satisfies the following:

\[ cF(z) + zF'(z) = (c + 1) f(z). \] \hspace{1cm} (2)

Other operators of interest include the Jung-Kim-Srivastava operators introduced in [9]. These operators defined as follows:

\[ P^v f(z) = \frac{2^v}{z \Gamma(v)} \int_0^z \left( \log \frac{z}{t} \right)^{v-1} f(t) \, dt \quad (v > 0) \]

\[ = z + \sum_{n=2}^{\infty} \left( \frac{2}{n + 1} \right)^v a_n z^n \]

and

\[ \ell_\mu^\nu f(z) = \left( \frac{v + \mu}{\mu} \right) \frac{v}{z^\mu} \int_0^z \left( 1 - \frac{t}{z} \right)^{v-1} t^{\mu-1} f(t) \, dt \quad (v > 0, \mu > -1) \]

\[ = z + \frac{\Gamma(v + \mu + 1)}{\Gamma(\mu + 1)} \sum_{n=2}^{\infty} \left( \frac{\Gamma(\mu + n)}{\Gamma(\nu + \mu + n)} \right) a_n z^n \]

have been studied by many authors (see [6, 16 and 24]).

In 1996, Sokol and Stankiewics [18] introduced the class \( SL^* \) consisting of normalized analytic functions \( f \) in \( D \) satisfying the condition

\[ \left| \frac{zf'(z)}{f(z)} \right|^2 - 1 < 1, \quad z \in D. \] Geometrically, a function \( f \) is in \( SL^* \) if \( \frac{zf'(z)}{f(z)} \)
lies in the interior of the right half of the lemniscate of Bernoulli
\[(x^2 + y^2)^2 - 2(x^2 - y^2) = 0\]. A function in the class \(SL^*\) is called a Sokol-Stankiewics starlike function. Alternatively, it can be established that \(f \in SL^* \iff \frac{zf''(z)}{f(z)} \prec \sqrt{1 + z}\), see [19]. Properties of functions in class \(SL^*\) have been studied by [1, 19-22].

Next, we denote \(S^*[A, B]\) as the class of Janowski starlike functions introduced by Janowski [8] and consisting of functions \(f \in A\) satisfying
\[
\frac{zf''(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1).
\]
Using the Dziok-Srivastava operator, new classes are formed. Classes denoted by \(SL^*[\alpha_1]\), \(H(\alpha_1; A, B; \lambda) (\lambda \in (0, 1])\) and \(CL[\alpha_1]\) are introduced and defined below:

\[
SL^*[\alpha_1] := \left\{ f : f \in S, \frac{z[H^{l,m}[\alpha_1]f(z)]'}{H^{l,m}[\alpha_1]f(z)} \prec \sqrt{1 + z}, z \in D \right\},
\]

\[
H(\alpha_1; A, B; \lambda) := \left\{ f : f \in S, \frac{z[H^{l,m}[\alpha_1]f(z)]'}{H^{l,m}[\alpha_1]f(z)} \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\lambda}, z \in D \right\},
\]

\[
CL[\alpha_1] := \left\{ f : f \in S, 1 + \frac{z[H^{l,m}[\alpha_1]f(z)]''}{[H^{l,m}[\alpha_1]f(z)]'} \prec \sqrt{1 + z}, z \in D \right\}.
\]

Quite trivially, the Alexander’s Theorem is also observed for the \(CL[\alpha_1]\) and \(SL^*[\alpha_1]\)
\[
f \in CL[\alpha_1] \iff zf''(z) \in SL^*[\alpha_1]. \quad (3)
\]

The authors in [4, 10, 11 and 13] established properties of functions in classes defined by various other operators using differential subordination. In
a similar manner, having defined the above classes using the Dziok-Srivastava operator, this paper uses differential subordination to determine inclusion theorems for functions to be in the classes $SL^*[\alpha_1]$, $H(\alpha_1; A, B; \lambda)$ and $CL[\alpha_1]$.

2. Main Results

In proving our results, the following lemmas will be required.

Lemma 2.1 [15]. Let $h$ be convex in $D$, with $Re[h(z) + \gamma] > 0$. If $p$ is analytic in $D$ with $p(0) = h(0)$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma 2.2 [10]. Let $\lambda \in (0, 1]$ be fixed, $\beta, \sigma \in C$, $\arg \beta \in \left(\frac{(1 - \lambda)\pi}{2}, \frac{\pi}{2}\right)$ and $\Re \sigma \geq 0$. Let $p$ be analytic function such that $p(0) = 1$ and $p(z) \neq -\frac{\sigma}{\beta}$ ($z \in D$). If

$$\left| \arg \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \sigma} \right\} \right| < \frac{\lambda \pi}{2},$$

then $|\arg p(z)| < \frac{\lambda \pi}{2}$.

Theorem 2.1. Let $\alpha_1 \geq 1$ and $\Re \{(\alpha_1 - 1) + \sqrt{1 + z}\} > 0$. Then $SL^*[\alpha_1 + 1] \subset SL^*[\alpha_1]$.

Proof. If $f \in SL^*[\alpha_1 + 1]$, then

$$\frac{z[H^{\lambda,m}[\alpha_1 + 1,f(z)]'}{H^{\lambda,m}[\alpha_1 + 1,f(z)]} \prec \sqrt{1 + z}.$$
and from (1), we have
\[
\frac{\alpha_1 H_{l,m}^{l,m}[\alpha_1 + 1] f(z)}{H_{l,m}^{l,m}[\alpha_1] f(z)} = z \left( \frac{H_{l,m}^{l,m}[\alpha_1] f(z)}{H_{l,m}^{l,m}[\alpha_1] f(z)} \right)' + (\alpha_1 - 1).
\]

Also, differentiating and rearranging give
\[
\frac{z[H_{l,m}^{l,m}[\alpha_1 + 1] f(z)]'}{H_{l,m}^{l,m}[\alpha_1 + 1] f(z)} = \frac{z[H_{l,m}^{l,m}[\alpha_1] f(z)]'}{H_{l,m}^{l,m}[\alpha_1] f(z)} + \left( \frac{z[H_{l,m}^{l,m}[\alpha_1] f(z)]'}{H_{l,m}^{l,m}[\alpha_1] f(z)} \right)' + \frac{z[H_{l,m}^{l,m}[\alpha_1] f(z)]'}{H_{l,m}^{l,m}[\alpha_1] f(z)} + \alpha_1 - 1.
\]

Letting \( p(z) = \frac{z[H_{l,m}^{l,m}[\alpha_1] f(z)]'}{H_{l,m}^{l,m}[\alpha_1] f(z)} \) and \( h(z) = \sqrt{1 + z} \), it is clear that \( h \) is convex in \( D \) and \( p(0) = h(0) \).

Since \( \frac{z[H_{l,m}^{l,m}[\alpha_1 + 1] f(z)]'}{H_{l,m}^{l,m}[\alpha_1 + 1] f(z)} < \sqrt{1 + z} \), applying Lemma 2.1 with \( \beta = 1 \) and \( \gamma = \alpha_1 - 1 \) proves \( \frac{z[H_{l,m}^{l,m}[\alpha_1] f(z)]'}{H_{l,m}^{l,m}[\alpha_1] f(z)} < \sqrt{1 + z} \). Thus \( f \in SL^*[\alpha_1] \).

**Theorem 2.2.** If for \( z \in D \), \( \Re\{c + \sqrt{1 + z}\} > 0 \) and \( f \in SL^*[\alpha_1] \), then \( F \in SL^*[\alpha_1] \).

**Proof.** Using the relation in (2), it can be established that
\[
cH_{l,m}^{l,m}[\alpha_1] F(z) + z[H_{l,m}^{l,m}[\alpha_1] F(z)]' = (c + 1) H_{l,m}^{l,m}[\alpha_1] f(z),
\]
which upon rewriting gives
\[
\frac{(c + 1) H_{l,m}^{l,m}[\alpha_1] f(z)}{H_{l,m}^{l,m}[\alpha_1] F(z)} = \frac{z[H_{l,m}^{l,m}[\alpha_1] F(z)]'}{H_{l,m}^{l,m}[\alpha_1] F(z)} + c.
\]
Differentiating both sides in the above equation and using the hypothesis that
$f \in SL^*[\alpha_1]$, we have the following relation

$$\frac{z[H^{l,m}[\alpha_1] f(z)]'}{H^{l,m}[\alpha_1] f(z)} = \frac{z[H^{l,m}[\alpha_1] F(z)]'}{H^{l,m}[\alpha_1] F(z)}$$

$$+ \frac{z \left( z[H^{l,m}[\alpha_1] F(z)]' \right)'}{H^{l,m}[\alpha_1] F(z)} < \sqrt{1 + z}. \tag{6}$$

As before, by letting $p(z) = \frac{z[H^{l,m}[\alpha_1] F(z)]'}{H^{l,m}[\alpha_1] F(z)}$ and $h(z) = \sqrt{1 + z}$. Lemma 2.1 implies $\frac{z[H^{l,m}[\alpha_1] F(z)]'}{H^{l,m}[\alpha_1] F(z)} < \sqrt{1 + z}$ and this completes the proof.

**Theorem 2.3.** Suppose for $z \in D$, $\text{Re} \{1 + \sqrt{1 + z}\} > 0$ and $P^{\nu-1} f(z) \in SL^*[\alpha_1]$. Then $P^{\nu} f(z) \in SL^*[\alpha_1](\nu > 1)$.

**Proof.** It can be derived

$$z[H^{l,m}[\alpha_1] P^{\nu} f(z)]' = 2[H^{l,m}[\alpha_1] P^{\nu-1} f(z)] - H^{l,m}[\alpha_1] P^{\nu} f(z)$$

and rearranging the equation gives

$$\frac{z[H^{l,m}[\alpha_1] P^{\nu} f(z)]'}{H^{l,m}[\alpha_1] P^{\nu} f(z)} = \frac{2[H^{l,m}[\alpha_1] P^{\nu-1} f(z)]}{H^{l,m}[\alpha_1] P^{\nu} f(z)} - 1.$$ 

Differentiating both sides, we obtain

$$\frac{z[H^{l,m}[\alpha_1] P^{\nu-1} f(z)]'}{H^{l,m}[\alpha_1] P^{\nu-1} f(z)} = p(z) + \frac{zp'(z)}{p(z) + 1} < \sqrt{1 + z},$$

where $p(z) = \frac{z[H^{l,m}[\alpha_1] P^{\nu} f(z)]'}{H^{l,m}[\alpha_1] P^{\nu} f(z)}$. Using Lemma 2.1 with $\beta = \gamma = 1$ implies the result.
Theorem 2.4. Let $\Re\{(\nu + \mu - 1) + \sqrt{1 + z}\} > 0$ for $\nu > 1$ and $\mu > -1$. \(\ell_{\mu}^{-1} f(z) \in SL^*[\alpha_1]\), then $\ell_{\mu}^{\nu} f(z) \in SL^*[\alpha_1]$. 

Proof. \(H^l, m[\alpha_1] \ell_{\mu}^{\nu} f(z) = (\nu + \mu)[H^l, m[\alpha_1] \ell_{\mu}^{-1} f(z)]\) 

\[-(\nu + \mu - 1)H^l, m[\alpha_1] \ell_{\mu}^{\nu} f(z),\]

From the differentiation of the above equation, we have

\[
\frac{z[H^l, m[\alpha_1] \ell_{\mu}^{\nu} f(z)]'}{H^l, m[\alpha_1] \ell_{\mu}^{\nu} f(z)} = \frac{(\nu + \mu)[H^l, m[\alpha_1] \ell_{\mu}^{-1} f(z)]}{H^l, m[\alpha_1] \ell_{\mu}^{\nu} f(z)} - (\nu + \mu - 1)\]

The hypothesis of theorem and Lemma 2.1 give the result by letting $p(z) = \frac{zp'(z)}{p(z) + (\nu + \mu - 1)}$. 

Theorem 2.5. Let $\lambda \in (0, 1]$ and $\Re(\alpha_1 - 1) \geq 0$, Then $H(\alpha_1 + 1; A, B; \lambda) \subset H(\alpha_1; A, B; \lambda)$. 

Proof. The proof is trivial. Since the Dziok-Srivastava operator satisfies (4) and $f \in H(\alpha_1 + 1; A, B; \lambda)$ implies

\[
\left|\Arg\left(\frac{z[H^l, m[\alpha_1 + 1] f(z)]'}{H^l, m[\alpha_1 + 1] f(z)}\right)\right| < \frac{\lambda \pi}{2},
\]

Therefore applying Lemma 2.2 with $p(z) = \frac{z[H^l, m[\alpha_1] f(z)]'}{H^l, m[\alpha_1] f(z)}$, $\beta = 1$ and...
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\[ \sigma = \alpha_1 - 1 \] gives

\[
| \arg p(z) | = \left| \arg \left( \frac{z[H^{1,m}[\alpha_1] f(z)]}{H^{1,m}[\alpha_1] f(z)} \right) \right| < \frac{\lambda \pi}{2}.
\]

Hence \( f \in H(\alpha_1; A, B; \lambda) \).

**Theorem 2.6.** Suppose \( \lambda \in (0, 1] \) and \( \text{Re } c \geq 0 \). If \( f \in H(\alpha_1; A, B; \lambda) \), then \( F \in H(\alpha_1; A; B; \lambda) \).

**Proof.** The proof follows easily since \( F \) satisfies equation (5) and with

\[
p(z) = \frac{z[H^{1,m}[\alpha_1] F(z)]}{H^{1,m}[\alpha_1] F(z)}\]

in Lemma 2.2 (\( \beta = 1 \) and \( \sigma = c \)) results

\[
| \arg p(z) | = \left| \arg \left( \frac{z[H^{1,m}[\alpha_1] F(z)]}{H^{1,m}[\alpha_1] F(z)} \right) \right| < \frac{\lambda \pi}{2}.
\]

**Remark 2.1.** In a similar manner as in previous theorems, it can easily be shown that \( P^{\nu-1} f(z) \in H(\alpha_1; A, B; \lambda) \Rightarrow P^{\nu} f(z) \in H(\alpha_1; A, B; \lambda) \) and \( \ell^{\nu-1} f(z) \in H(\alpha_1; A, B; \lambda) \Rightarrow \ell^{\nu} f(z) \in H(\alpha_1; A, B; \lambda) \).

**Remark 2.2.** For \( \lambda = \frac{1}{2}, A = 1 \) and \( B = 0 \), Theorem 2.5 and Theorem 2.6 reduce to Theorem 2.1 and Theorem 2.2.

**Theorem 2.7.** Let \( \alpha_1 \geq 1 \). Then \( \text{CL} [\alpha_1 + 1] \subset \text{CL} [\alpha_1] \).

**Proof.** Using (3) and Theorem 2.1 will easily deduce our result.

\[
f(z) \in \text{CL} [\alpha_1 + 1]
\]

\[
\Leftrightarrow z f'(z) \in \text{SL}^*[\alpha_1 + 1]
\]

\[
\Rightarrow z f'(z) \in \text{SL}^*[\alpha_1] \Leftrightarrow H^{1,m}[\alpha_1] [zf'(z)]' \in \text{SL}^*
\]

\[
\Leftrightarrow z [H^{1,m}[\alpha_1] f(z)]' \in \text{SL}^* \Leftrightarrow H^{1,m}[\alpha_1] f(z) \in \text{CL}
\]

\[
\Leftrightarrow f \in \text{CL} [\alpha_1].
\]
Theorem 2.8. If \( f \in CL[\alpha_1] \), then \( F \in CL[\alpha_1] \).

Proof. By applying Theorem 2.2, it follows that
\[
f \in CL[\alpha_1] \iff zf''(z) \in SL^*[\alpha_1]
\]
\[
\Rightarrow F[zf''(z)] \in SL^*[\alpha_1] \iff z[F(z)]' \in SL^*[\alpha_1]
\]
\[
\Leftrightarrow F \in CL[\alpha_1].
\]

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References


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