Multivalent Harmonic Functions Defined by Dziok-Srivastava Operator

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Abstract. Necessary and sufficient coefficient bounds and convolution condition for certain multivalent harmonic functions whose convolution with generalized hypergeometric functions is starlike of order $\gamma$ are investigated. Results on extreme points, convex combination and distortion bounds using the coefficient condition are also obtained.

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1. Introduction

A complex-valued continuous function $f = u + iv$ in a complex domain $E \subset \mathbb{C}$ is said to be harmonic if both $u$ and $v$ are real harmonic in $E$. There is an interrelation between harmonic functions and analytic functions. If $E$ is a simply connected domain, then $f = h + \bar{g}$ where $h$ and $g$ are analytic in $E$; the functions $h$ and $g$ are respectively called the analytic part and co-analytic part of $f$. The function $f = h + \bar{g}$ is said to be harmonic univalent in $E$ if the mapping $z \rightarrow f(z)$ is orientation preserving, harmonic and univalent in $E$. This mapping is orientation preserving and locally univalent in $E$ if and only if the Jacobian $J_f$ of $f$ given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ is positive in $E$ [16]. From the perspective of geometric functions theory, Clunie and Sheil-Small [10] initiated the study on these functions by introducing the class $S_H$ consisting of normalized complex-valued harmonic univalent functions $f$ defined on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. They gave necessary and sufficient conditions for $f$ to be locally univalent and sense-preserving in $\mathbb{D}$. Coefficient bounds for functions in $S_H$ were obtained. Since then, various subclasses of $S_H$ were investigated by several authors [1, 5, 8, 9, 15, 19, 20, 21]. Note that the class $S_H$ reduces to the class of normalized analytic univalent functions if the co-analytic part of $f$ is identically to zero ($g \equiv 0$).

Multivalent harmonic functions in $\mathbb{D}$ were introduced by Duren, Hengartner and Lauge-Hansen [11] via the argument principle. In [2], the class of multivalent harmonic functions and the class $S_H^*(p, \gamma)$ of multivalent harmonic starlike functions of order $\gamma$, where $p \geq 1$,
0 \leq \gamma < 1 \) were discussed and studied. Motivated by [4], we introduce a class of multivalent harmonic functions starlike of order \( \gamma \) using the Dziok-Srivastava operator. Several related work using other linear operators can also be found in [3, 14, 26, 24].

Recall that the convolution of two analytic functions \( \varphi(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( \psi(z) = \sum_{n=0}^{\infty} b_n z^n \) defined on \( \mathbb{D} \) is the analytic function given by \( \varphi(z) * \psi(z) = \sum_{n=0}^{\infty} a_n b_n z^n = \psi(z) * \varphi(z) \). Let \( S_H(p) \) denote the class of multivalent harmonic functions \( f = h + \bar{g} \) where

\[
(1.1) \quad h(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1}.
\]

For \( \alpha \in \mathbb{C} \ (i = 1, 2, \ldots, l) \) and \( \beta_j \in \mathbb{C} \backslash \{0, -1, -2, \ldots\} \) \( (j = 1, 2, \ldots, m) \), the generalized hypergeometric function \( _pF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \) is given by

\[
_{p}F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n n!} z^n \quad (l \leq m + 1; \ l, m \in N_0 := N \cup \{0\}; z \in D)
\]

where \((\lambda)_n\) is the Pochhammer symbol defined, in terms of gamma function, by

\[
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \ \lambda \neq 0 \\ \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1), & n = 1, 2, 3, \ldots \end{cases}
\]

For an analytic function \( h \) of the form (1.1), Dziok and Srivastava [12] introduced the linear operator

\[
H_p^m(\alpha_1) h(z) = z^p \ _pF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * h(z)
\]

which includes well known operators such as the Hohlov operator [13], Carlson-Shaffer operator [7], Ruscheweyh derivative operator [22], the generalized Bernardi-Libera-Livingston integral operator [6], [17], [18] and the Srivastava-Owa fractional derivative operator [25]. For a harmonic function \( f = h + \bar{g} \), with \( h \) and \( g \) given by (1.1), the Dziok-Srivastava operator is defined by

\[
H_p^m(\alpha_1) f(z) = H_p^m(\alpha_1) h(z) + \overline{H_p^m(\alpha_1) g(z)},
\]

where \( H_p^m(\alpha_1) h(z) = z^p + \sum_{n=2}^{\infty} \phi_n a_{n+p-1} z^{n+p-1} \), \( H_p^m(\alpha_1) g(z) = \sum_{n=1}^{\infty} \phi_n b_{n+p-1} z^{n+p-1} \) and

\[
\phi_n = \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n (n - 1)!},
\]

\( \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m \) are positive real numbers such that \( l \leq m + 1 \).

Denote by \( S_H^\gamma(p, \alpha_1, \gamma) \), the class of multivalent harmonic functions satisfying

\[
(1.3) \quad \Re \left( \frac{z \left( H_p^m(\alpha_1) h(z) \right) - z \left( H_p^m(\alpha_1) g(z) \right)'}{(H_p^m(\alpha_1) h(z) + \overline{H_p^m(\alpha_1) g(z)})} \right) \geq p \gamma
\]

for \( p \geq 1, \ 0 \leq \gamma < 1, \ |z| = r < 1 \). Note that \( S_H^\gamma(1, \alpha_1, \gamma) \equiv S_H^\gamma(\alpha_1, \gamma) \) is the class defined in [4]. In the case of \( l = m + 1 \) and \( \alpha_1 = \beta_1, \ldots, \alpha_l = \beta_m \), \( S_H^\gamma(p, 1, \gamma) \equiv S_H^\gamma(p, \gamma) \) is investigated in [2] and \( S_H^\gamma(1, 1, \gamma) \equiv S_H^\gamma(\gamma) \) is the class introduced by Jahangiri [15]. Further
$T^*_H(p, \alpha_1, \gamma)$, $p \geq 1$ denotes the class of functions $f = h + \bar{g} \in S^*_H(p, \alpha_1, \gamma)$ where $h$ and $g$ are functions of the form

$$h(z) = z^p - \sum_{n=2}^{\infty} |a_{n+p-1}|z^{n+p-1}, \quad g(z) = \sum_{n=1}^{\infty} |b_{n+p-1}|z^{n+p-1}.$$ (1.4)

2. Main results

Necessary coefficient conditions for the harmonic starlike functions and harmonic convex functions can be found in [10] and [23]. Now we derive sufficient coefficient bound for the class $S^*_H(p, \alpha_1, \gamma)$.

**Theorem 2.1.** Let $f = h + \bar{g}$ be given by (1.1) and $\prod_{i=1}^{l} (\alpha_i)_{n-1} \geq \prod_{j=1}^{n} (\beta_j)_{n-1} (n-1)!$. If

$$\sum_{n=2}^{\infty} \left( \frac{n + p (1 - \gamma) - 1}{p (1 - \gamma)} |a_{n+p-1}| + \frac{n + p (1 + \gamma) - 1}{p (1 - \gamma)} |b_{n+p-1}| \right) |\phi_n| \leq 1 - \frac{1 + \gamma}{1 - \gamma} |b_p|$$ (2.1)

where $|b_p| < (1 - \gamma)/(1 + \gamma)$, $0 \leq \gamma < 1$ and $\phi_n$ is given by (1.2), then the harmonic function $f$ is orientation preserving in $\mathbb{D}$ and $f \in S^*_H(p, \alpha_1, \gamma)$.

**Proof.** The inequality $|h'(z)| \geq |g'(z)|$ is enough to show that $f$ is orientation preserving. Note that

$$|h'(z)| \geq p |z|^{p-1} - \sum_{n=2}^{\infty} (n + p - 1) |a_{n+p-1}| |z|^{n+p-2}$$

$$= p |z|^{p-1} \left( 1 - \sum_{n=2}^{\infty} \frac{(n + p - 1)}{p} |a_{n+p-1}| |z|^{n-1} \right)$$

$$\geq p |z|^{p-1} \left( 1 - \sum_{n=2}^{\infty} \frac{(n + p - 1)}{p} |a_{n+p-1}| \right)$$

$$\geq |z|^{p-1} \left( 1 - \sum_{n=2}^{\infty} \frac{(n + p (1 - \gamma) - 1)}{p (1 - \gamma)} |\phi_n| |a_{n+p-1}| \right)$$

By hypothesis, since $|\phi_n| \geq 1$ and by (2.1),

$$|h'(z)| \geq |z|^{p-1} \left( \frac{1 + \gamma}{1 - \gamma} |b_p| + \sum_{n=2}^{\infty} \frac{(n + p (1 + \gamma) - 1)}{p (1 - \gamma)} |\phi_n| |b_{n+p-1}| \right)$$

$$= |z|^{p-1} \left( \sum_{n=1}^{\infty} \frac{(n + p (1 + \gamma) - 1)}{p (1 - \gamma)} |\phi_n| |b_{n+p-1}| \right)$$

$$\geq |z|^{p-1} \left( \sum_{n=1}^{\infty} (n + p - 1) |b_{n+p-1}| \right)$$

$$\geq |z|^{p-1} \left( \sum_{n=1}^{\infty} (n + p - 1) |b_{n+p-1}| |z|^{n-1} \right)$$

$$= \sum_{n=1}^{\infty} (n + p - 1) |b_{n+p-1}| |z|^{n+p-2} = |g'(z)|$$

Thus, $f$ is orientation preserving in $\mathbb{D}$. 
Next, we prove $f \in S'_H(p, \alpha_1, \gamma)$ by establishing condition (1.3). First, let

$$w(z) = z \left( H^m_p[\alpha_1]h(z) \right)' - z \left( H^m_p[\alpha_1]g(z) \right)' = \frac{A(z)}{B(z)},$$

where

$$A(z) = z \left( H^m_p[\alpha_1]h(z) \right)' - z \left( H^m_p[\alpha_1]g(z) \right)' = \frac{B(z)}{z^{n+1}}.$$  

Now,

$$|A(z) + p (1 - \gamma) B(z)| - |A(z) - p (1 + \gamma) B(z)|$$

$$\geq (2p - p \gamma) |z^p| - \sum_{n=2}^{\infty} (n + 2p - p \gamma - 1) |\phi_n a_{n+1} z^{n+1}|$$

$$- \sum_{n=1}^{\infty} (n + p \gamma - 1) |\phi_n b_{n+1} a_{n+1} z^{n+1}| - p \gamma |z^p|$$

$$\geq 2p (1 - \gamma) |z^p| \left( 1 - \sum_{n=2}^{\infty} (n + p - p \gamma - 1) |\phi_n a_{n+1}||z^{n+1}| \right)$$

$$\geq 2p (1 - \gamma) |z^p| \left( 1 - \sum_{n=2}^{\infty} (n + p - p \gamma - 1) |\phi_n a_{n+1}||z^{n+1}| \right)$$

$$\geq 2p (1 - \gamma) |z^p| \left( 1 - \sum_{n=1}^{\infty} \left( \frac{n + p - p \gamma - 1}{p (1 - \gamma)} \right) |\phi_n a_{n+1}| \right)$$

$$= 2p (1 - \gamma) |z^p| \left( 1 - \sum_{n=1}^{\infty} \left( \frac{n + p - p \gamma - 1}{p (1 - \gamma)} \right) |\phi_n| a_{n+1} \right)$$

$$+ \left( \frac{n + p - p \gamma - 1}{p (1 - \gamma)} |\phi_n| a_{n+1} \right)$$

The last expression is non-negative by (2.1). Since $\text{Re } w \geq p \gamma$ if and only if $|A(z) + p (1 - \gamma) B(z)| \geq |A(z) - p (1 + \gamma) B(z)|$, $f \in S'_H(p, \alpha_1, \gamma)$.

For $\sum_{n=1}^{\infty} \left( |a_{n+1}| + |\gamma_{n+1}| \right) = 1$ and $\gamma_p = 0$, the function

$$f_1(z) = z^p + \sum_{n=2}^{\infty} \frac{p (1 - \gamma)}{n + p (1 - \gamma) - 1} |\phi_n| a_{n+1} z^{n+1} + \sum_{n=1}^{\infty} \frac{p (1 - \gamma)}{n + p (1 + \gamma) - 1} |\phi_n| \gamma_{n+1} z^{n+1}$$
shows equality in the coefficient bound given by (2.1). For the function $f_1$ defined in (2.2), the coefficients are

$$a_{n+p-1} = \frac{p(1 - \gamma)}{[n + p(1 - \gamma) - 1]|\phi_n|} x_{n+p-1} \quad \text{and} \quad b_{n+p-1} = \frac{p(1 - \gamma)}{[n + p(1 + \gamma) - 1]|\phi_n|} y_{n+p-1},$$

and since condition (2.1) holds, this implies $f_1 \in S^*_H(p, \alpha_1, \gamma)$.

To show that the converse need not be true, consider the function

$$f(z) = z^p + \frac{p(1 - \gamma)}{[1 + p(1 - \gamma)]|\phi_2|} z^{p+1} + \frac{\gamma - 1}{2(1 + \gamma)} z^p.$$

It can be shown that

$$\Re\left( \frac{z}\left[ \frac{p(1 - \gamma)}{[1 + p(1 - \gamma)]|\phi_2|} z^{p+1} - \frac{\gamma - 1}{2(1 + \gamma)} z^p \right] \right) \geq p\gamma, \quad (p \geq 1, 0 \leq \gamma < 1)$$

thus $f \in S^*_H(p, \alpha_1, \gamma)$ but

$$\sum_{n=2}^{\infty} \frac{n + p(1 - \gamma) - 1}{p(1 - \gamma)} |a_{n+p-1}| |\phi_n| + \sum_{n=1}^{\infty} \frac{n + p(1 + \gamma) - 1}{p(1 - \gamma)} |b_{n+p-1}| |\phi_n|$$

$$= \frac{1 + p(1 - \gamma)}{p(1 - \gamma)} \left| \frac{p(1 - \gamma)}{[1 + p(1 - \gamma)]|\phi_2|} \right| |\phi_2| + \frac{1 + \gamma}{1 - \gamma} \frac{\gamma - 1}{2(1 + \gamma)} > 1.$$

The next result provide a convolution condition for $f$ to be in the class $S^*_H(p, \alpha_1, \gamma)$.

**Theorem 2.2.** $f \in S^*_H(p, \alpha_1, \gamma)$ if and only if

$$H^l_{p} [\alpha_1] h(z) * \left[ \frac{2p(1 - \gamma)z^p + (\xi - 2p + 2p\gamma + 1)z^{p+1}}{(1 - z)^2} \right]$$

$$- H^l_{p} [\alpha_1] g(z) * \left[ \frac{2p(\xi + \gamma)z^p + (\xi - 2p\xi - 2p\gamma + 1)z^{p+1}}{(1 - z)^2} \right] \neq 0, \quad |\xi| = 1, z \in D.$$

**Proof.** A necessary and sufficient condition for $f \in S^*_H(p, \alpha_1, \gamma)$ is given by (1.3) and we have

$$\Re\left( \frac{1}{p(1 - \gamma)} \left[ z \left( H^l_{p} [\alpha_1] h(z) \right)' - z \left( H^l_{p} [\alpha_1] g(z) \right)' \right] - p\gamma \right) \geq 0.$$

Since

$$\frac{1}{p(1 - \gamma)} \left[ z \left( H^l_{p} [\alpha_1] h(z) \right)' - z \left( H^l_{p} [\alpha_1] g(z) \right)' \right] - p\gamma$$

$$= \frac{1}{p(1 - \gamma)} \left[ p + \sum_{n=2}^{\infty} (n + p - 1) \phi_n a_{n+p-1} z^{n-1} - \frac{p}{2p} \sum_{n=1}^{\infty} (n + p - 1) \phi_n b_{n+p-1} z^{n-1} \right] - p\gamma$$

$$= 1$$
at \( z = 0 \), the above required condition is equivalent to
\[
\frac{1}{p(1-\gamma)} \left[ z \left( H_p^l, m [\alpha_1] h(z) \right)' - z \left( H_p^l, m [\alpha_1] g(z) \right)' - p\gamma \right] = \frac{\xi - 1}{\xi + 1},
\]
Simple algebraic manipulation in (2.3) yields
\[
0 \neq (\xi + 1) \left( z \left( H_p^l, m [\alpha_1] h(z) \right)' - z \left( H_p^l, m [\alpha_1] g(z) \right)' - p\gamma H_p^l, m [\alpha_1] h(z) - p\gamma H_p^l, m [\alpha_1] g(z) \right) - (\xi - 1)p(1-\gamma)H_p^l, m [\alpha_1] h(z) - (\xi - 1)p(1-\gamma)H_p^l, m [\alpha_1] g(z)
\]
\[
= H_p^l, m [\alpha_1] h(z) \sum_{m=1}^p \left( \frac{z^p}{(1-z)^2} - \frac{(1-p)z^p}{(1-z)} \right) - (2p\gamma + p\xi - p)z^p
\]
\[
= H_p^l, m [\alpha_1] h(z) \sum_{m=1}^p \left( \frac{2p(1-\gamma)z^p + (\xi - 2p + 2p\gamma + 1)z^{p+1}}{(1-z)^2} \right)
\]
The coefficient bound for class \( T_H^+(p, \alpha_1, \gamma) \) is determined in the following theorem. Furthermore, we use the coefficient condition to obtain extreme points, convex combination and distortion bounds.

**Theorem 2.3.** Let \( f = h + \bar{g} \) be given by (1.4). Then \( f \in T_H^+(p, \alpha_1, \gamma) \) if and only if
\[
\sum_{n=2}^{n+p} \left( \frac{n+p}{p(1-\gamma)} - 1 \right) \left| a_{n+p-1} \right| + \frac{n+p}{p(1-\gamma)} \left| b_{n+p-1} \right| \phi_n \leq 1 - \frac{1+\gamma}{1-\gamma} \left| b_p \right|
\]
where \( \left| b_p \right| < (1-\gamma)/(1+\gamma) \), \( 0 < \gamma < 1 \) and \( \phi_n \) is given by (1.2).

**Proof.** Since \( T_H^+(p, \alpha_1, \gamma) \subset S_H^+(p, \alpha_1, \gamma) \), sufficiency part follows from Theorem 2.1. To prove the necessity part, suppose that \( f \in T_H^+(p, \alpha_1, \gamma) \). Then we obtain
\[
\Re \left( \frac{pz^n - \sum_{n=2}^{n+p-1} (n+p-1) |a_{n+p-1}| |\phi_n|^{n+p-1} - \sum_{n=1}^{n+p-1} (n+p-1) |\phi_n|^{n+p-1} |\bar{b_{n+p-1}}| \phi_n z^{n+p-1}}{z^n - \sum_{n=2}^{n+p} |a_{n+p-1}| |\phi_n|^{n+p-1} + \sum_{n=1}^{n+p-1} |\phi_n|^{n+p-1} |\bar{b_{n+p-1}}| \phi_n z^{n+p-1}} \right) \geq p\gamma,
\]
and the result follows by letting \( r \rightarrow 1^- \) along real axis.

Let \( \text{clco} \ T_H^+(p, \alpha_1, \gamma) \) denote the closed convex hull of \( T_H^+(p, \alpha_1, \gamma) \). Now we determine the extreme points of \( \text{clco} \ T_H^+(p, \alpha_1, \gamma) \).

**Theorem 2.4.** Let \( f \) be given by (1.4). Then \( f \in \text{clco} \ T_H^+(p, \alpha_1, \gamma) \) if and only if \( f \) can be expressed in the form
\[
f = \sum_{n=1}^{n+p} (X_{n+p-1} h_{n+p-1} + Y_{n+p-1} g_{n+p-1}),(2.5)
\]
where
\[ h_p(z) = z^p, \quad h_{n+p-1}(z) = z^p - \frac{p(1-\gamma)}{n+p(1-\gamma)-1} |\phi_n| z^{n+p-1} \quad (n = 2, 3, \ldots), \]
\[ g_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{n+p(1+\gamma)-1} |\phi_n| z^{n+p-1} \quad (n = 1, 2, 3, \ldots), \]
\( \phi_n \) is given by (1.2), and \( \sum_{n=1}^{\infty} (X_{n+p-1} + Y_{n+p-1}) = 1, \) with \( X_{n+p-1} \geq 0, Y_{n+p-1} \geq 0. \) In particular, the extreme points of \( T_{\mathbb{H}}(p, \alpha_1, \gamma) \) are \( h_{n+p-1} \) and \( g_{n+p-1}. \)

**Proof.** Let \( f \) be of the form (2.5), then we have
\[
f(z) = X_p h_p(z) + \sum_{n=2}^{\infty} X_{n+p-1} \left( z^p - \frac{p(1-\gamma)}{n+p(1-\gamma)-1} |\phi_n| z^{n+p-1} \right) \]
\+
\[
+ \sum_{n=1}^{\infty} Y_{n+p-1} \left( z^p + \frac{p(1-\gamma)}{n+p(1+\gamma)-1} |\phi_n| z^{n+p-1} \right) \]
\[
= z^p - \sum_{n=2}^{\infty} \frac{p(1-\gamma)}{n+p(1-\gamma)-1} |\phi_n| X_{n+p-1} z^{n+p-1} \]
\+
\[
+ \sum_{n=1}^{\infty} \frac{p(1-\gamma)}{n+p(1+\gamma)-1} |\phi_n| Y_{n+p-1} z^{n+p-1}. \]
(2.6)

Furthermore, let
\[
|a_{n+p-1}| = \frac{p(1-\gamma)}{n+p(1-\gamma)-1} |\phi_n| X_{n+p-1} \quad \text{and} \quad |b_{n+p-1}| = \frac{p(1-\gamma)}{n+p(1+\gamma)-1} |\phi_n| Y_{n+p-1}. \]

Then
\[
\sum_{n=2}^{\infty} \frac{n+p(1-\gamma)-1}{p(1-\gamma)} |\phi_n| \left( \frac{p(1-\gamma)}{n+p(1-\gamma)-1} |\phi_n| X_{n+p-1} \right) \]
\+
\[
+ \sum_{n=1}^{\infty} \frac{n+p(1+\gamma)-1}{p(1-\gamma)} |\phi_n| \left( \frac{p(1-\gamma)}{n+p(1+\gamma)-1} |\phi_n| Y_{n+p-1} \right) \]
\[
= \sum_{n=2}^{\infty} X_{n+p-1} + \sum_{n=1}^{\infty} Y_{n+p-1} = 1 - X_p \leq 1. \]

Thus \( f \in \text{clco} \ T_{\mathbb{H}}(p, \alpha_1, \gamma). \)

Conversely, suppose that \( f \in \text{clco} \ T_{\mathbb{H}}(p, \alpha_1, \gamma). \) Set
\[
X_{n+p-1} = \frac{n+p(1-\gamma)-1}{p(1-\gamma)} |\phi_n| |a_{n+p-1}| \quad (n = 2, 3, \ldots), \]
\[
Y_{n+p-1} = \frac{n+p(1+\gamma)-1}{p(1-\gamma)} |\phi_n| |b_{n+p-1}| \quad (n = 1, 2, \ldots), \]

and define \( X_p = 1 - \sum_{n=2}^{\infty} X_{n+p-1} - \sum_{n=1}^{\infty} Y_{n+p-1}. \) Then,
\[
f(z) = z^p - \sum_{n=2}^{\infty} |a_{n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| z^{n+p-1} \]
\[
= z^p - \sum_{n=2}^{\infty} \frac{p(1-\gamma)X_{n+p-1}}{n+p(1-\gamma)-1} |\phi_n| z^{n+p-1} + \sum_{n=1}^{\infty} \frac{p(1-\gamma)Y_{n+p-1}}{n+p(1+\gamma)-1} |\phi_n| z^{n+p-1}. \]
Then, by (2.7)

\[ \begin{align*}
= & X_p z^p + \sum_{n=2}^{\infty} X_{n+p-1} \left( z^n - \frac{p (1-\gamma)}{n+p (1-\gamma) - 1} |\phi_n| z^{n+p-1} \right) \\
+ & \sum_{n=1}^{\infty} Y_{n+p-1} \left( z^n + \frac{p (1-\gamma)}{n+p (1+\gamma) - 1} |\phi_n| z^{n+p-1} \right) \\
= & \sum_{n=1}^{\infty} (X_{n+p-1} h_{n+p-1} + Y_{n+p-1} g_{n+p-1}) \\
\end{align*} \]

as required.

**Theorem 2.5.** The class \( T_H^+(p, \alpha_1, \gamma) \) is closed under convex combination.

**Proof.** For \( i = 1, 2, 3, \ldots \), suppose that \( f_i(z) \in T_H^+(p, \alpha_1, \gamma) \), where \( f_i \) is given by

\[ f_i(z) = z^n - \sum_{n=2}^{\infty} |a_{i,n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{i,n+p-1}| z^{n+p-1}. \]

By Theorem 2.3,

\[ \sum_{n=2}^{\infty} \frac{n+p (1-\gamma) - 1}{p (1-\gamma)} |\phi_n| |a_{i,n+p-1}| + \sum_{n=1}^{\infty} \frac{n+p (1+\gamma) - 1}{p (1-\gamma)} |\phi_n| |b_{i,n+p-1}| \leq 1. \]

For \( \sum_{i=1}^{\infty} t_i = 1 \), \( 0 \leq t_i \leq 1 \), the convex combination of \( f_i \) may be written as,

\[ \sum_{i=1}^{\infty} t_i f_i(z) = z^n - \sum_{i=1}^{\infty} \left( \sum_{n=2}^{\infty} |a_{i,n+p-1}| \right) z^{n+p-1} + \sum_{i=1}^{\infty} \left( \sum_{n=1}^{\infty} |b_{i,n+p-1}| \right) z^{n+p-1}. \]

Then, by (2.7)

\[ \begin{align*}
\sum_{n=2}^{\infty} \frac{n+p (1-\gamma) - 1}{p (1-\gamma)} |\phi_n| & \left( \sum_{i=1}^{\infty} t_i |a_{i,n+p-1}| \right) \\
+ & \sum_{n=1}^{\infty} \frac{n+p (1+\gamma) - 1}{p (1-\gamma)} |\phi_n| \left( \sum_{i=1}^{\infty} t_i |b_{i,n+p-1}| \right) \\
= & \sum_{i=1}^{\infty} t_i \left( \sum_{n=2}^{\infty} \frac{n+p (1-\gamma) - 1}{p (1-\gamma)} |\phi_n| |a_{i,n+p-1}| + \sum_{n=1}^{\infty} \frac{n+p (1+\gamma) - 1}{p (1-\gamma)} |\phi_n| |b_{i,n+p-1}| \right) \\
\leq & \sum_{i=1}^{\infty} t_i = 1.
\end{align*} \]

Hence, \( \sum_{i=1}^{\infty} t_i f_i(z) \in T_H^+(p, \alpha_1, \gamma) \).

In the last theorem below we give distortion inequalities for \( f \) in the class \( T_H^+(p, \alpha_1, \gamma) \).

**Theorem 2.6.** If \( f \in T_H^+(p, \alpha_1, \gamma) \) with \( \phi_n \geq \phi_2 \), then for \( |z| = r < 1 \),

\[ |f(z)| \leq (1 + |b_p|) r^p + r^{p+1} \left( \frac{p (1-\gamma)}{p (1-\gamma) + 1} |\phi_2| - \frac{p (1+\gamma) |b_p|}{p (1+\gamma) + 1} \phi_2 \right) \]

and

\[ |f(z)| \geq (1 - |b_p|) r^p - r^{p+1} \left( \frac{p (1-\gamma)}{p (1-\gamma) + 1} |\phi_2| - \frac{p (1+\gamma) |b_p|}{p (1+\gamma) + 1} \phi_2 \right). \]
Proof. Since

\[ \frac{p (1 - \gamma) + 1}{p (1 - \gamma)} |\phi_2| \sum_{n=2}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \]
\[ \leq \sum_{n=2}^{\infty} \frac{n + p (1 - \gamma) - 1}{p (1 - \gamma)} (|a_{n+p-1}| + |b_{n+p-1}|) |\phi_a| \]
\[ \leq \sum_{n=2}^{\infty} \left( \frac{n + p (1 - \gamma) - 1}{p (1 - \gamma)} |a_{n+p-1}| + \frac{n + p (1 + \gamma) - 1}{p (1 - \gamma)} |b_{n+p-1}| \right) |\phi_a|, \]

the result of Theorem 2.3 gives

\[ \sum_{n=2}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \leq \frac{p (1 - \gamma)}{[p (1 - \gamma) + 1] |\phi_2|} \left( 1 - \frac{1 + \gamma}{1 - \gamma} |b_p| \right). \]  

(2.8)

Next, again since \( f \in T_H^*(p, \alpha_1, \gamma) \), we have from (2.8) and \(|z| = r\) that

\[ |f(z)| = |z|^p - \sum_{n=2}^{\infty} |a_{n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| \bar{z}^{n+p-1} \]
\[ \leq |z|^p + \sum_{n=2}^{\infty} |a_{n+p-1}| |z|^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| |\bar{z}|^{n+p-1} \]
\[ = r^p + \sum_{n=2}^{\infty} |a_{n+p-1}| r^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| r^{n+p-1} \]
\[ \leq (1 + |b_p|) r^p + \left( \sum_{n=2}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \right) r^{p+1} \]
\[ \leq (1 + |b_p|) r^p + r^{p+1} \left( \frac{p (1 - \gamma)}{[p (1 - \gamma) + 1] |\phi_2|} - \frac{p (1 + \gamma) |b_p|}{[p (1 - \gamma) + 1] |\phi_2|} \right) \]

which gives the first result.

In a similar manner, we obtain the following lower bound.

\[ |f(z)| \geq r^p - \sum_{n=2}^{\infty} |a_{n+p-1}| r^{n+p-1} - \sum_{n=1}^{\infty} |b_{n+p-1}| r^{n+p-1} \]
\[ = (1 - |b_p|) r^p - \sum_{n=2}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) r^{n+p-1} \]
\[ \geq (1 - |b_p|) r^p - r^{p+1} \left( \frac{p (1 - \gamma)}{[p (1 - \gamma) + 1] |\phi_2|} - \frac{p (1 + \gamma) |b_p|}{[p (1 - \gamma) + 1] |\phi_2|} \right). \]

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References


