Optimal integrated policies for a single-vendor single-buyer time-varying demand model

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\textbf{A B S T R A C T}

This paper presents a procedure for finding the optimal replenishment and production schedule for a single-vendor single-buyer inventory model and where the objective is to minimize the total integrated inventory costs of the vendor and the buyer over one production schedule and a finite-planning horizon. The production rate of the vendor is assumed fixed and the demand rate of the buyer is assumed to take some general form and is a function of time. It is shown that for a fixed number of replenishment schedules, \( n \), the optimal times of ordering are unique and can be found as a solution of some system of nonlinear equations. These in turn give the optimal order quantities of the buyer in each period. Moreover, the optimal value function is shown to be convex in \( n \).

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1. Introduction

This paper proposes a method for solving optimally the replenishment and the production schedule for an inventory model with a single-vendor single-buyer and time-varying demand and where the objective is to minimize the integrated inventory costs over a finite-planning horizon. The planning horizon contains one production run and may contain several replenishment schedules.

The interest in inventory models of the type considered in this paper is driven by the search for efficient management inventory strategies across a supply chain by aligning and coordinating activities in order to improve the performance of the supply chain. This coordination may take the form of sharing information, resources, costs, and possibly profit. Examples of such coordination is already present in the automotive industry; see [1].

The model treated in this paper was considered in [2] for linearly decreasing demand and equal lots policy of shipment from the vendor to the buyer. Earlier works on this model are found in [3–6]. For more details see Omar [2]. Nevertheless, the problem of finding the optimal replenishment and the production schedule for the buyer and the vendor respectively remained open. The existing work revolved around suggesting heuristics for some models and no attempt for developing a general theory for tackling this problem is known.

The objective of this paper is to suggest a procedure for solving this outstanding problem. In doing so, the demand rates of earlier models are allowed to take a general form. The proposed procedure is drawn from a general theory developed by Benkherouf and Gilding [7] for finite horizon inventory models. Although, the theory was developed for classical inventory models, it turns out that it is equally applicable (with a slight modification), as we shall see, to the model of this paper (to be presented below). The basic idea is to consider the integrated costs function for the buyer and the vendor for a fixed number of replenishment periods. Direct computations then show that this cost function has a structure which renders the

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analysis possible through the theory of Benkherouf and Gilding [7]. The integrated costs function can be shown to possess a unique minimum under some mild technical conditions. Moreover, the value function of the integrated costs is convex in the number of replenishment periods.

This paper is organized as follows. In the next section the model is introduced as well as the notation used in this paper. Section 3 is concerned with preparing the ground for the presentation of the optimal procedure. Section 4 presents preliminaries on the theory of Benkherouf and Gilding [7] which are needed to tackle the problem. Section 5 contains the optimal procedure and the paper concludes with numerical examples and some general remarks.

2. The model

We shall use the word vendor and supplier interchangeably.

Consider an inventory model which consists of controlling the level of stock of a single product over a known and finite-planning horizon of length $H$, where $H > 0$. Below are the assumptions of the model:

1. The demand rate for the product $D(t)$ is a function of time $t$. The function is assumed to be continuously differentiable on the interval $[0, H]$.
2. The production rate is fixed to $p > 0$. The manufacturer has a basket containing the stock needed to be consumed in period 1.
3. Production is stopped on the interval $(t_1, H)$. At time $H$, production is restarted again and a new period is initiated. The buyer policy for acquiring goods is to decide on the his (her) ordering policy on the interval $(0, H)$.
4. Setup cost of production is fixed and is denoted by $A_1 > 0$.
5. Set up cost for shipment is fixed and is denoted by $A_2 > 0$.
6. Inventory holding cost for the vendor is $c_1 > 0$.
7. Inventory holding cost for the buyer is $c_2 > 0$, where $c_2 > c_1$.

Throughout this paper we shall use the following notation:

- $\partial_t$: The derivative of a univariate function.
- $\partial_x$: The partial derivative of a bivariate function with respect to the first variable.
- $\partial_y$: The partial derivative of a bivariate function with respect to the second variable.
- $\partial_x^2$: The second partial derivative of a bivariate function with respect to the first variable.
- $\partial_y^2$: The second partial derivative of a bivariate function with respect to the second variable.
- $\partial_x\partial_y$: The cross partial derivative of a bivariate function.
- $C(\Xi)$: The space of continuous functions over a set $\Xi$.
- $C^1(\Xi)$: The space of continuously differentiable functions over a set $\Xi$.

Assume that the initial inventory for both buyer and supplier is zero.

Let us assume, for simplicity, that for a single production cycle the total time to consume the amount produced is $H$. We also assume that at time $t = 0$, the manufacturer has a basket containing the stock needed to be consumed in period 1.

On the interval $[0, H]$, the vendor produces goods from time $t = 0$ to time $t = T_p (0 < T_p < H)$, at a rate $p$. Then, production is stopped on the interval $(T_p, H)$. At time $H$, production is restarted again and a new period is initiated. The buyer order for acquiring goods is to decide on the his (her) ordering policy on the interval $(0, H)$. We suppose that the manufacturer makes $n$ orders at times $0 = t_0 < t_1 < \cdots < t_n = H$, where $n$ and $T_p = (t_0, t_1, \ldots, t_n)$ are to be determined. In period 1, the buyer order is delivered from the basket and part of the production in period 1 goes to refilling the basket while the rest of the production is consumed in the subsequent periods. Refilling the basket entails a fixed cost in addition to the variable cost.

We shall examine the buyer and the manufacturer separately. We start with the buyer. Fig. 1 shows a typical change in the level of stock.

Let $I(t)$ denote the level of stock of the buyer at time $t$.

The dynamics of the inventory level for the buyer, in the ordering period $i$, is governed by the differential equation

$$I'_i(t) = -D(t), \quad t_{i-1} < t < t_i, \quad \text{with } I_0(t) \uparrow 0, \quad \text{as } t \uparrow t_i. \quad (2.1)$$

Direct computations show that the total inventory cost for the buyer, $CB$, is

$$CB = nA_2 + c_2 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_{i}} (t - t_{i-1})D(t)dt, \quad (2.2)$$

with

$$I_i(t) = \int_{t_{j+1}}^{t_{j}} D(t)dt, \quad j = 0, \ldots, n - 1. \quad (2.3)$$

Let $k$ be an index referring to the replenishment periods during the production period such that $t_k \leq T_p$, and $t_{k+1} > T_p$. If $y$ refers to the level of stock of the manufacturer at time $t$. It follows that the dynamics of the inventory level, in the absence of ordering, for the manufacturer is governed by

$$y'(t) = p, \quad \text{for } t \leq T_p. \quad (2.4)$$
and for $t > T_p$, impulses of sizes $I(t_{k+1}), I(t_{k+2}), \ldots$ corresponding to the replenishment of the buyer at times $t_{k+1}, t_{k+2}, \ldots$ (respectively) bring the level of stock $y(t)$ down.

Fig. 2 shows a typical change in the level of stock for the manufacturer. Define

$$IR(t_i) = \sum_{j=0}^{i} I_b(t_j), \quad i = 1, \ldots, n - 1,$$

and $IR(t_0) = 0$, where $I_b(t_j)$ is given by (2.3).

It can be shown that

$$IR(t_i) = \int_{0}^{t_{i+1}} D(t) \, dt, \quad i = 1, \ldots, n - 1. \quad (2.6)$$

Again, direct computations show that for $t \leq T_p$, the inventory level for the manufacturer is given by

$$y(t) = \begin{cases} pt, & \text{for } 0 \leq t < \min\{T_p, t_1\} \\ pt - IR(t_{i-1}), & \text{for } t_{i-1} \leq t < T_p, \quad i = 2, \ldots, n - 1. \end{cases} \quad (2.7)$$

It follows that the amount of inventory for the manufacturer in period $i$, $(i < k)$, is given by

$$\int_{t_{i-1}}^{t_i} (pt - IR(t_{i-1})) \, dt, \quad (2.8)$$

when $T_p \geq t_1$, and for $T_p < t_1$, this is equal to

$$\int_{0}^{T_p} pt \, dt + \int_{T_p}^{t_1} pT_p \, dt. \quad (2.9)$$

For $t > T_p$, production is stopped. Therefore the amount of inventory in period $i$ $(i > k)$, is given by

$$\int_{t_{i-1}}^{t_i} (pT_p - IR(t_{i-1})) \, dt. \quad (2.10)$$

Period $k$ requires special treatment as production is stopped at this period. In this period the amount of inventory is given by

$$\int_{t_k}^{T_p} (pt - IR(t_k)) \, dt + \int_{T_p}^{t_{k+1}} (pT_p - IR(t_k)) \, dt. \quad (2.11)$$

Gathering the costs altogether gives that for the manufacturer this is equal to

$$CM := A_1 + c_1 \left\{ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (pt - IR(t_{i-1})) \, dt + \int_{t_k}^{T_p} (pt - IR(t_k)) \, dt + \int_{T_p}^{t_{k+1}} (pT_p - IR(t_k)) \, dt \right. \\
+ \left. \sum_{i=k+2}^{n} \int_{t_{i-1}}^{t_i} (pT_p - IR(t_{i-1})) \, dt \right\}. \quad (2.12)$$
The total cost, $T_{C_n}$, for both buyer and manufacturer is given by

$$T_{C_n} = CB + CM,$$

where $CB$ is given by (2.2) and $CM$ is given by (2.12).

The problem is then to find $n, T_p, t_1, \ldots, t_n$ which minimizes $T_{C_n}$.

3. Modeling preliminaries

We assume that the total amount of goods produced during the time $H$ is either consumed or goes to the manufacturer basket. This implies that $T_p$ is fixed and is given by

$$T_p = \frac{1}{p} \int_0^H D(t) \, dt.$$

Also, the expression below pertaining to $CM$ in (2.12)

$$\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (pt - IR(t_{i-1})) \, dt + \int_{t_k}^{T_p} (pt - IR(t_k)) \, dt + \int_{T_p}^{t_k+1} (pT_p - IR(t_k)) \, dt + \sum_{i=k+2}^{n} \int_{t_{i-1}}^{t_i} (pT_p - IR(t_{i-1})) \, dt,$$

is equal to

$$\int_0^{T_p} pt \, dt - \left\{ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} IR(t_{i-1}) \, dt + \int_{t_k}^{T_p} IR(t_k) \, dt \right\} + \int_{T_p}^{H} pT_p \, dt - \left\{ \int_{T_p}^{t_k+1} IR(t_k) \, dt + \sum_{i=k+2}^{n} \int_{t_{i-1}}^{t_i} IR(t_{i-1}) \, dt \right\}.$$

Therefore, by (2.12) the total cost of the manufacturer can be written as

$$CM = A_1 + c_1 \left\{ \int_0^{T_p} pt \, dt + \int_{T_p}^{H} pT_p \, dt - \sum_{i=2}^{n} \int_{t_{i-1}}^{t_i} IR(t_{i-1}) \, dt \right\}.$$

This is equal to

$$K_1 - c_1 \sum_{i=2}^{n} \int_{t_{i-1}}^{t_i} IR(t_{i-1}) \, dt,$$

where

$$K_1 = A_1 + c_1 \left\{ \int_0^{T_p} pt \, dt + \int_{T_p}^{H} pT_p \, dt \right\},$$

is a known constant.
It can be shown that
\[ \sum_{i=2}^{n} \int_{t_{i-1}}^{t_i} IR(t_{i-1}) \, dt = \sum_{i=2}^{n} (t_i - t_{i-1}) \int_{0}^{t_i} D(t) \, dt \]
\[ = -t_1 \int_{0}^{t_1} D(t) \, dt - \sum_{i=2}^{n-1} t_i \left\{ \int_{0}^{t_i} D(t) \, dt - \int_{0}^{t_{i+1}} D(t) \, dt \right\} + t_n \int_{0}^{t_n} D(t) \, dt, \]
\[ = -t_1 \int_{0}^{t_1} D(t) \, dt - \sum_{i=2}^{n} t_{i-1} \int_{t_{i-1}}^{t_i} D(t) \, dt + H \int_{0}^{t_n} D(t) \, dt. \]

Set
\[ K_1 = \mathcal{K}_1 - c_1 H \int_{0}^{H} D(t) \, dt. \]

It follows that
\[ CM = K_1 + c_1 \left\{ t_1 \int_{0}^{t_1} D(t) \, dt + \sum_{i=2}^{n} t_{i-1} \int_{t_{i-1}}^{t_i} D(t) \, dt \right\}. \]

Consequently, by (2.13) and using (2.2) the total cost \( CB + CM \) is given by
\[ TC_n = K_1 + nA_2 + c_2 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) D(t) \, dt + c_1 \left\{ t_1 \int_{0}^{t_1} D(t) \, dt + \sum_{i=2}^{n} t_{i-1} \int_{t_{i-1}}^{t_i} D(t) \, dt \right\}. \]

Now use the fact
\[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} t \, D(t) \, dt, \]
is fixed to infer that minimizing \( TC_n \) reduces to minimizing \( C_n \) which is given below
\[ C_n = nA_2 + \int_{0}^{t_1} \left[ c_2 t + c_1 (t_1 - t) \right] D(t) \, dt + \left[ \sum_{i=2}^{n} \int_{t_{i-1}}^{t_i} (c_2 - c_1)(t - t_{i-1}) D(t) \, dt \right]. \]

**Remark 1.** Note that in (3.2) although the total inventory cost is dependent on the production rate \( p \), the optimal replenishment schedule is independent of \( p \), which seems remarkable. This will turn out to be crucial in applying the theory of Benkherouf and Gilding to this particular problem. Also, note that the total cost of the inventory is given by \( TC_n \) in (3.1) and not \( C_n \).

**4. Technical preliminaries**

Recall that the objective is to find \( n \) and \((t_0, t_1, \ldots, t_n)\) which minimizes \( C \) given (3.2) subject to the constraints
\[ 0 = t_0 < t_1 < t_2 < \cdots < t_n = H. \]

The resulting optimization problem is amenable to analysis by a theory developed by Benkherouf and Gilding [7] for finite horizon models. Finite horizon inventory models with time-varying demand were treated in [8–15].

We shall briefly outline the main ingredients of the theory which shall be needed to solve our optimization problem.

The general problem treated in [7] and adapted to our case considers finding \( n \) and \((t_0, t_1, \ldots, t_n)\) which minimizes a function \( C_n \) given by
\[ C_n = nA_2 + \sum_{i=1}^{n} R_i(t_{i-1}, t_i), \]
subject to constraints (4.1).

In our case
\[ R_1(x, y) = \int_{x}^{y} \left[ c_2 t + c_1 (y - t) \right] D(t) \, dt, \]
and
\[ R_i(x, y) = \int_{x}^{y} (c_2 - c_1)(t - x) D(t) \, dt, \quad i = 2, \ldots, n. \]

Note that \( R_i(x, y) = R_j(x, y) \) for \( i \geq 2 \) and \( j \geq 2 \). The functions \( R_i \)'s, for all \( i \geq 1 \), were assumed in [7] to be defined on the set \( \Omega = \{(x, y) : 0 \leq x < y \leq H\} \),

and satisfies the generic hypothesis...
Hypothesis 1. For every $i \geq 1$, the function $R_i$ is twice differentiable on $\Omega$ and its one sided derivatives exist on the boundary of $\Omega$, and for $(x, y) \in \Omega$ the function $R_i$ is such that
\begin{align*}
R_i(x, y) &> 0, \quad \text{for } y > x, \quad (4.5) \\
R_i(x, x) &= 0, \quad (4.6) \\
(\partial_x R_i)(x, y) &< 0 < (\partial_y R_i)(x, y), \quad (4.7)
\end{align*}
and
\begin{equation}
(\partial_x \partial_y R_i)(x, y) < 0. \quad (4.8)
\end{equation}

An additional hypothesis to Hypothesis 1 was required for the theory to go through.

Hypothesis 2. For $(x, y) \in \Omega$, there holds
\begin{equation}
\partial_y R_i + \partial_x R_{i+1} = 0 \quad \text{on } \hat{\Omega} \setminus \Omega, \quad (4.9)
\end{equation}
for all $1 \leq i \leq n - 1$. Moreover, there is a function $f \in C(0, H)$ such that
\begin{equation}
\mathcal{L}_x R_{i+1}(x, y) \geq 0 \quad \text{and} \quad \mathcal{L}_y R_i(x, y) \geq 0, \quad \text{in } \Omega, \quad (4.10)
\end{equation}
for all $1 \leq i \leq n - 1$, where
\begin{equation}
\mathcal{L}_x z = \partial_y^2 z + \partial_x z + f(x) \partial_x z, \quad (4.11)
\end{equation}
and
\begin{equation}
\mathcal{L}_y z = \partial_y \partial_x z + \partial_y^2 z + f(y) \partial_y z. \quad (4.12)
\end{equation}

Define
\begin{equation}
S_n(t_0, \ldots, t_n) = \sum_{i=1}^{n} R_i(t_i, t_{i-1}). \quad (4.13)
\end{equation}

The next theorem, from [7], shows that under assumptions Hypotheses 1 and 2, the function $S_n$ has a unique minimum.

Theorem 1. The function $S_n$ given by (4.13) has a unique minimum with respect to $t_0, t_1, \ldots, t_n$ satisfying (4.1).

Benkherouf and Gilding established the existence of a sequence of functions $t_i \subset C([0, H]) \cap C^1(0, H), i = 1, \ldots, n - 1$, which gives the optimal solution of $S_n$, such that
\begin{equation}
t_i(0) = 0, \quad (4.14)
\end{equation}
and for $0 \leq \eta \leq H$
\begin{equation}
0 < t_i'(\eta) < 1, \quad (4.15)
\end{equation}
and
\begin{equation}
t_i = t_i(t_{i+1}) \quad \text{for } i = n - 1, n - 2, \ldots, 0. \quad (4.16)
\end{equation}
Furthermore, if $s_n(h)$ refers to the minimum value of $S_n$ on $[0, h]$ then
\begin{equation}
s_n(h) = (\partial_x R_n)(t_{n-1}(h), h). \quad (4.17)
\end{equation}

Benkherouf and Gilding [7] showed that the optimal solution $t_1, \ldots, t_{n-1}$ is the unique solution of the nonlinear system of equations
\begin{equation}
(\partial_y R_i)(t_{i-1}, t_i) + (\partial_x R_{i+1})(t_i, t_{i+1}) = 0, \quad i = 1, \ldots, n - 1. \quad (4.18)
\end{equation}

This system may be solved recursively using the sequence $(t_i)$. Indeed for $i = n - 1$ (4.18) leads to
\begin{equation}
(\partial_y R_{n-1})(t_{n-2}, t_{n-1}) + (\partial_x R_n)(t_{n-1}, H) = 0. \quad (4.19)
\end{equation}

If $t_{n-1}$ is known, then $t_{n-2}$ can be found uniquely as a function of $t_{n-1}$ and consequently $t_{n-2} = t_{n-2}(t_{n-1})$. Moreover, $t_{n-2}$ is increasing in $t_{n-1}$. This process is extended to find the remaining $t_i's, i = n - 3, \ldots, 0$, as a function of $t_{n-1}$. More details on this process will be discussed below.

The next theorem can be found in [7].

Theorem 2. If $R_i = R, \text{for all } i \geq 1$, then the function $s_n$ is convex in $n$.

Theorems 1 and 2 play a crucial role in finding the optimal values of $n$ with corresponding $t_1, \ldots, t_n$. This will be carried out in the next section.
5. Optimal solution

Recall the definition of $R_i$ in (4.4)

$$R_1(x, y) = \int_x^y \{c_2 t + c_1(y - t)\} D(t)dt,$$

and for $i \geq 2$,

$$R_i(x, y) = \int_x^y (c_2 - c_1)(t - x)D(t)dt, \quad i = 2, \ldots, n. \tag{5.2}$$

Direct computations show that

$$\begin{align*}
(\partial_y R_1)(x, y) &= -\{c_2 x + c_1(y - x)\} D(x), \\
(\partial_y R_1)(x, y) &= c_2 y D(y) + \int_x^y c_1 D(t)dt, \\
(\partial_y \partial_y R_1)(x, y) &= -c_1 D(x), \\
(\partial_y^2 R_1)(x, y) &= (c_2 - c_1)D'(x) - [c_2 x + c_1(y - x)] D'(x), \\
(\partial_y^2 R_1)(x, y) &= (c_2 + c_1)D(y) + c_2 y D'(y),
\end{align*}$$

for $i \geq 2$

$$\begin{align*}
(\partial_x R_i)(x, y) &= -(c_2 - c_1) \int_x^y D(t)dt, \\
(\partial_x R_i)(x, y) &= (c_2 - c_1)(y - x)D(y), \\
(\partial_x \partial_x R_i)(x, y) &= -(c_2 - c_1) D(y), \\
(\partial_x^2 R_i)(x, y) &= (c_2 - c_1)D(x), \\
(\partial_x^2 R_i)(x, y) &= (c_2 - c_1)(y - x)D'(y) + (c_2 - c_1)D(y).
\end{align*}$$

It is easy to show that for $(x, y) \in \Omega$, and $i = 1, \ldots, n$, the function $R_i$ satisfies Hypothesis 1.

Assume first that $n$ is fixed and consider minimizing $S_n$ given by (4.13). Setting the first partial derivatives to zero reduces to

$$\begin{align*}
(\partial_y R_i)(t_{i-1}, t_i) + (\partial_y R_{i+1})(t_i, t_{i+1}) = 0, \quad i = 1, \ldots, n - 1. \tag{5.13}
\end{align*}$$

The next theorem shows that if the demand rate is logconcave then Hypothesis (4.10) of Hypothesis 2 is satisfied.

**Theorem 3.** If the function $D'/D$ is nonincreasing, then Hypothesis (4.10) of Hypothesis 2 is satisfied.

**Proof.** The key idea in the proof is to be able to find an appropriate $f$ that makes (4.10) hold. Set

$$f(x) = \frac{D'(x)}{D(x)}. \tag{5.14}$$

Consider first the case $i \geq 2$. Using (4.11) with (5.8)–(5.12) leads to

$$\begin{align*}
\mathcal{L}_x R_i(x, y) &= (c_2 - c_1) \left[\{D(x) - D(y)\} - f(x) \int_x^y D(t)dt\right] \\
&= (c_2 - c_1) \left[\{D(x) - D(y)\} + \frac{D'(x)}{D(x)} \int_x^y D(t)dt\right].
\end{align*}$$

Under the assumption that $c_2 > c_1$, we have $\mathcal{L}_x R(x, y) \geq 0$ is equivalent to

$$\frac{D'(x)}{D(x)} \geq \frac{D(y) - D(x)}{\int_x^y D(t)dt}. \tag{5.15}$$

The extended mean value theorem shows that the right-hand side of (5.15) is equal to $D'(\xi)/D(\xi)$ for some $\xi \in (x, y)$. Therefore $\mathcal{L}_x R_i(x, y) \geq 0$, since $\frac{D}{D}$ is nonincreasing for $i \geq 2$.

Now, we turn to $\mathcal{L}_y R_i(x, y)$. Again, using (4.12) with (5.8)–(5.12) leads to

$$\begin{align*}
\mathcal{L}_y R_i(x, y) &= (c_2 - c_1)(y - x)D'(y) + f(y)(c_2 - c_1)(y - x)D(y) \\
&= 0.
\end{align*}$$
To complete the proof we need to check that \( \mathcal{L} \phi R_1(x, y) \geq 0 \). We have by (5.4), (5.5) and (5.7) to
\[
\mathcal{L} \phi R_1(x, y) = (c_2 + c_1)D(y) + c_2yD'(y) - c_1D(x) + f(y) \left[ c_2yD(y) + \int_x^y c_1D(t)dt \right],
\]
or
\[
\mathcal{L} \phi R_1(x, y) = c_2D(y) + c_1 \left[ (D(y) - D(x)) - \frac{D'(y)}{D(y)} \int_x^y D(t)dt \right] \geq c_1 \left[ (D(y) - D(x)) - \frac{D'(y)}{D(y)} \int_x^y D(t)dt \right].
\]
Using the extended mean value theorem shows that the right-hand side of the last inequality is non-negative. This completes the proof. \( \square \)

The class of demand functions \( D \) with \( D'/D \) nonincreasing includes the linear as well as the exponential rate functions. It is easy to check that for \( i = 2, \ldots, n-1 \), \( R_i \) given by (5.2) satisfies (4.9) of Hypothesis 2. However, when \( i = 1 \), (5.2) is not always satisfied. Indeed, we have
\[
\begin{align*}
\partial_t R_1(0, 0) + \partial_x R_2(0, 0) &= 0, \\
\partial_t R_1(h, h) + \partial_x R_2(h, h) &> 0, \quad \text{for } 0 < h \leq H.
\end{align*}
\]
At first sight it may appear that the theory of Benkherouf and Gilding will not be applicable to the present model. Fortunately, going back to the details of the proof of Theorem 1 (the key theorem), and others, in [7] shows that the results of paper [7] are still valid and therefore applicable to the present case. Details are omitted here since they are technical and are essentially present in [7]. Therefore, the next result is stated without proof.

**Corollary 1.** The function \( S_n \) defined in (4.13) and with \( R_i \) given by (4.3) and (4.4) has a unique minimum which is the solution of the system of nonlinear equations given by (5.13).

**Lemma 1.** If \( t_1, \ldots, t_{n-1} \) is the optimal solution of \( S_n \), then for \( i = 2, \ldots, n-1 \),

(i) If \( D \) is increasing then
\[
t_{i+1} - t_i < t_i - t_{i-1}, \quad \text{and} \quad t_1 > \left( \frac{c_2 - c_1}{c_1 + c_2} \right) (t_2 - t_1).
\]

(ii) If \( D \) is decreasing then
\[
t_{i+1} - t_i > t_i - t_{i-1}, \quad \text{and} \quad t_1 < \left( \frac{c_2 - c_1}{c_1 + c_2} \right) (t_2 - t_1).
\]

(iii) If \( D \) is constant then
\[
t_{i+1} - t_i = t_i - t_{i-1}, \quad \text{and} \quad t_1 = \left( \frac{c_2 - c_1}{c_1 + c_2} \right) (t_2 - t_1).
\]

**Proof.** We shall only prove (i). The proof of (ii) is similar to that of (i) and (iii) is easy to obtain.

The optimal solution \( t_1, \ldots, t_{n-1} \), by (5.13), satisfies for \( i = 2, \ldots, n-2 \)
\[
(t_i - t_{i-1})D(t_i) = \int_{t_{i-1}}^{t_i} D(t)dt. \tag{5.16}
\]
If the function \( D \) is increasing, then the right-hand side of (5.16) is greater than
\[
(t_{i+1} - t_i)D(t_i).
\]
The rest of the proof is left as exercise. This completes the proof. \( \square \)

Theorem 1 sets a basis for computing the optimal values \( t_1, \ldots, t_{n-1} \). The system of nonlinear equations given by (5.16) plays a key factor in determining these values. To be precise, we have for \( i = n-2 \)
\[
(t_{n-1} - t_{n-2})D(t_{n-1}) = \int_{t_{n-2}}^{t_{n-1}} D(t)dt.
\]
Theorem 1 states that if \( t_{n-1} \) is known, then \( t_{n-2} \) can be uniquely found as a function of \( t_{n-1} \) such that \( t_{n-2} = \tau_{n-2} \). Furthermore, this function is increasing as a function of \( t_{n-1} \). Likewise, (5.16) for \( i = n-2 \), shows that \( t_{n-3} \) can be found as
a function of \( t_{n-2} \) and consequently is a function of \( t_{n-1} \), where \( t_{n-3} \) is increasing in \( t_{n-1} \). This process is iterated until \( i = 1 \), where it is required that \( t_0 = r_0(t_1) = 0 \). Also,

\[
c_2t_1D(t_1) + c_1 \int_0^{t_1} D(t)dt = (c_2 - c_1) \int_{t_1}^{t_2} D(t)dt.
\]

The function \( t_0 \) is increasing in \( t_{n-1} \), with \( t_0(H) > 0 \) and \( t_0(0) < 0 \). This implies that a univariate search for the root of the equation \( t_0(t_1) = 0 \) can be undertaken which guarantees a unique root.

Let \( s_n(H) \) correspond to the optimal value of \( S_n \). Theorem 2 implies that \( s_n \) is convex in subject to Hypotheses 1 and 2 be satisfied. This is not the case of the present model as we have encountered earlier. However, the convexity result will still hold to the model of the present here. The proof of convexity is similar to that in \([7,12,14,13]\).

Define \( s_{N+1}(h) \) to be the minimal value of \( S_{N+1}(h) \) for a model with time horizon of length \( h \). Then, using the dynamic programming principle we get

\[
s_{N+1}(h) = \min_{0 \leq \eta \leq h} V_{N+1}(\eta, h),
\]

where

\[
V_{N+1}(\eta, h) = s_N(\eta) + R_{N+1}(\eta, h).
\]

The next lemma is required for the proof of convexity.

**Lemma 2.** The sequence \( \{\tau_i\}, i = 1, \ldots, n - 1 \), defined in Theorem 1 satisfies

\[
\tau_{i-1}(\eta) \leq \tau_i(\eta), \quad \text{for all } 0 \leq \eta \leq H.
\]

**Proof.** The proof is by induction on the number of periods. If \( i = 1 \), the result is immediate. Assume that the result is true for \( i = N - 1 \geq 1 \) and let us prove that it is also true for \( i = N \). It follows from the definition of \( V \) in (5.18) and using (4.17) that

\[
(\partial_\eta V_{N+1})(\eta, h) = s'_N(\eta) + (\partial_\eta R_{N+1})(\eta, h)
\]

\[
= (\partial_\eta R_N)(\tau_{N-1}(\eta), \eta) + (\partial_\eta R_{N+1})(\eta, h).
\]

But (4.8) with the induction hypothesis we obtain that

\[
(\partial_\eta R_N)(\tau_{N-1}(\eta), \eta) < (\partial_\eta R_N)(\tau_{N-2}(\eta), \eta).
\]

Therefore

\[
(\partial_\eta V_{N+1})(\eta, h) < (\partial_\eta R_N)(\tau_{N-2}(\eta), \eta) + (\partial_\eta R_{N+1})(\eta, h).
\]

Set \( \eta = \tau_{N-1}(h) \), to get that the right-hand side of (5.20) is equal to

\[
(\partial_\eta R_N)(\tau_{N-2}(\tau_{N-1}(h)), \tau_{N-1}(h)) + (\partial_\eta R_{N+1})(\tau_{N-1}(h), h).
\]

This is equal to zero by (5.13). Hence \( (\partial_\eta V_{N+1})(\tau_{N-1}(h), h) < 0 \).

Now, recall that \( \tau_N(\eta) \) is the unique solution \( \eta \) such that \( (\partial_\eta V_{N+1})(\eta, h) = 0 \). Also, by (4.8)

\[
(\partial_\eta V_{N+1})(h, h) = (\partial_\eta R_N)(\tau_{N-1}(h), h) + (\partial_\eta R_{N+1})(h, h)
\]

\[
> (\partial_\eta R_N)(h, h) + (\partial_\eta R_{N+1})(h, h)
\]

\[
\geq 0.
\]

Therefore \( \tau_{N-1}(h) < \tau_N(h) \), and the proof is complete. \( \Box \)

**Proof of the Convexity of \( s_n \).** Before we finalize the proof we need to show that

\[
s'_n(H) - s'_{n+1}(H) > 0.
\]

Indeed, (4.17) shows that

\[
s'_n(H) - s'_{n+1}(H) = (\partial_\eta R_n)(\tau_{n-1}(H), H) - (\partial_\eta R_{n+1})(\tau_n(H), H).
\]

Assumption (4.8) and Lemma 2 lead to (5.21) for \( n = 1 \)

\[
s'_1(H) - s'_2(H) = (\partial_\eta R_1)(0, H) - (\partial_\eta R_2)(\tau_1(H), H)
\]

\[
> (\partial_\eta R_1)(\tau_1(H), H) - (\partial_\eta R_2)(\tau_1(H), H)
\]

\[
= [c_2H - (c_2 - c_1)(H - \tau_1(H))] D(H) + c_1 \int_{\tau_1(H)}^H D(t)dt > 0.
\]
For \( n = 2 \), (5.22) gives
\[
 s'_n(H) - s'_{n+1}(H) > (\partial_y R_n)(\tau_n(H), H) - (\partial_y R_{n+1})(\tau_n(H), H) = 0
\]
since \( R_n = R_{n+1} \) for \( n \geq 2 \).

Now, (5.18) gives
\[
 s_{n+1}(\tau_{n+1}(H)) - s_{n+2}(\tau_{n+2}(H)) = V_{n+1}(\tau_n(H), H) - V_{n+2}(\tau_n(H), H).
\]
But \( \tau_n(H) \) minimizes \( V_{n+1}(\eta, H) \), therefore the above expression is less than
\[
 V_{n+1}(\tau_{n+1}(H), H) - V_{n+2}(\tau_{n+1}(H), H).
\]
This is equal, using (5.18), to
\[
 s_{n+1}(\tau_{n+1}(H)) - s_{n+2}(\tau_{n+2}(H)) < 0,
\]
which is less than
\[
 s_{n+1}(H) - s_{n+2}(H),
\]
since \( \tau_{n+1}(H) < H \) and using (5.21). Whence \( s_n \) is convex. □

The next corollary summarizes the result for the optimal policy of our inventory model.

**Corollary 2.** The optimal number of replenishment schedule is such that

(i) If \( A_2 > s_1 - s_2 \) then the optimal number of replenishment schedule is \( n = 1 \).

(ii) If there exists an \( N \geq 2 \) such that \( s_N - s_N + 1 > A_2 > s_N - s_{N+1} \), then the optimal number of replenishment schedule is \( N \).

(iii) If there exists an \( N \geq 1 \) such that \( A_2 = s_N - s_{N+1} \), then there are two optimal number of replenishment schedules. These are \( N \) and \( N + 1 \).

**Lemma 3.** If \( D \equiv \text{constant} \) then the optimal value of \( C_n \) for \( n \geq 2 \), is
\[
 nA_2 + \frac{1}{2} (c_1 + c_2) t_1^2 + \frac{1}{2} (n - 1) \left( \frac{c_2 - c_1}{M_n} \right) H^2 D, \tag{5.23}
\]
where
\[
 M_n = (n - 1) + \frac{c_2 - c_1}{c_1 + c_2}, \tag{5.24}
\]
and
\[
 t_1 = \left( \frac{c_2 - c_1}{c_1 + c_2} \right) \frac{H}{M_n}.
\]

**Proof.** The key element in the proof is part (iii) of Lemma 1. Direct computations using the fact that
\[
 t_1 + \sum_{i=2}^{n-1} (t_i - t_{i-1}) = H,
\]
give that for \( i \geq 2 \)
\[
 t_i - t_{i-1} = \frac{H}{M_n},
\]
where \( M \) is given by (5.24), and
\[
 t_1 = \left( \frac{c_2 - c_1}{c_1 + c_2} \right) \frac{H}{M}.
\]
The rest of the proof is then immediate. □

6. Numerical examples and conclusions

This section presents four numerical examples of the model presented in Section 2. The linear demand rate and the exponential demand rate functions are considered when they are decreasing and increasing.
6.1. The linear demand rate

Let \( D(t) = a + bt \), then (5.2) gives
\[
R_1(x, y) = \frac{1}{6} \left( b(c_1 + 2c_2)y^3 + 3a(c_1 + c_2)y^2 - 3c_1x(2a + bx)y + (c_1 - c_2)x^2(3a + 2bx) \right),
\]
and
\[
R_i(x, y) = \frac{1}{6} (c_2 - c_1)(x - y)^2(3a + b(x + 2y)), \quad \text{for } i \geq 2.
\]
Let \( H = 5 \), \( a = 6 \), \( A_2 = 80 \). The first example corresponds to the increasing linear function demand rate and Example 2 below corresponds to the decreasing linear function demand rate.

Example 1. Let \( b = 5 \), \( c_1 = 10 \), and \( c_2 = 20 \). The optimal solution is \( n = 6 \) with \( t_1 = 0.522 \), \( t_2 = 1.642 \), \( t_3 = 2.601 \), \( t_4 = 3.463 \), \( t_5 = 4.252 \), \( t_6 = 5 \), and \( c_n = 913.1784 \).

Example 2. Let \( b = -1 \), \( c_1 = 10 \), and \( c_2 = 20 \). The optimal solution is \( n = 3 \) with \( t_1 = 0.484 \), \( t_2 = 2.234 \), \( t_3 = 5 \), and \( c_n = 400.2611 \).

6.2. The exponential demand rate

Let \( D(t) = a \exp(bt) \), then (5.2) gives
\[
R_1(x, y) = \frac{ae^{by}(-bc_2x + c_2 + c_1(b(x - y) - 1)) + ae^{by}(c_1 + c_2(by - 1))}{b^2},
\]
and
\[
R_i(x, y) = \frac{a(c_2 - c_1) \left( e^{by}(y - x) - 1 + e^{by} \right)}{b^2}, \quad \text{for } i \geq 2.
\]
Let \( H = 8 \), \( a = 1 \). The first example corresponds to the increasing demand rate functions and Example 3 below corresponds to the decreasing demand rate function.

Example 3. Let \( b = 1 \), \( c_1 = 10 \), \( A_2 = 100 \), and \( c_2 = 20 \). The optimal solution is \( n = 12 \) with \( t_1 = 1.993 \), \( t_2 = 3.759 \), \( t_3 = 4.777 \), \( t_4 = 5.479 \), \( t_5 = 6.011 \), \( t_6 = 6.437 \), \( t_7 = 6.792 \), \( t_8 = 7.096 \), \( t_9 = 7.362 \), \( t_{10} = 7.597 \), \( t_{11} = 7.808 \), \( t_{12} = 8 \), and \( c_n = 6769.2921 \).

Example 4. Let \( b = -1 \), \( c_1 = 10 \), \( A_2 = 20 \), and \( c_2 = 20 \). The optimal solution is \( n = 2 \) with \( t_1 = 0.315 \), \( t_2 = 8 \), and \( c_n = 91.1392 \).

Note that the results of the optimal replenishment schedules are in accordance with statements (i) and (ii) of Lemma 1. In this paper we presented a procedure for finding the optimal replenishment and production schedule for a single-vendor single-buyer inventory model and where the objective is to minimize the total integrated inventory costs of the vendor and the buyer over one production schedule and a finite-planning horizon. The production rate of the vendor is assumed fixed and the demand rate of the buyer is assumed to take some general form and is a function of time. It was shown that for a fixed number of replenishment schedule the optimal times of ordering are unique and can be found as a solution of some system of nonlinear equations. Moreover, the optimal value function was shown to be convex in the number of replenishment periods. Numerical examples were also presented.

It is worth noting that if \( c_1 = c_2 \), then the expression for \( c_n \) given by (4.2) reduces to
\[
c_n = nA_2 + R_1(0, t_1),
\]
where \( R_1(0, t_1) \) is given by (4.4), and \( c_1 = A_2 + R(0, H) \). It is clear that for \( n \geq 2 \), \( t_1 > 0 \), and the optimal solution is to make \( t_1 \) very small and \( n = 2 \). Then, the optimal solution is such that if
\[
2A_2 > A_2 + R(0, H),
\]
then \( n = 1 \), otherwise, \( n = 2 \) and \( t_1 \) is set to a small value.

It is the opinion of the authors that extensions of the procedure for models with deteriorating and (or) stock demand dependent items may be possible: see Goyal and Giri [16]. Also, it would be of interest to examine the case where \( c_2 < c_1 \). A case where the theory of Benkherouf and Gilding does not seem to be applicable.

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