An integrated equal-lots policy for shipping a vendor's final production batch to a single buyer under linearly decreasing demand

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Abstract

This paper considers a supply chain in which a vendor supplies a product to a buyer. The vendor is about to manufacture the final batch of the product at a finite rate and then periodically ship the output to the buyer. The buyer then consumes the product at a linearly decreasing demand rate. Most previous work on this topic has been based on the assumption of fixed demand rate. Costs are attached to the manufacturing batch setup, the delivery of a shipment, and stockholding at the vendor and buyer. The objective is to determine the number of shipments and sizes of those shipments which minimise the total cost—assuming the vendor and buyer collaborate and find a way of sharing the consequent benefits. We show how the optimal policy may be derived when the shipments size are identical. We illustrate this policy with numerical examples.

1. Introduction

Much attention has been paid in recent years to managed supply chains, partly as a consequence of the relentless drive to lower costs and partly as a consequence of the facility for parties in the chain to share information electronically. A common feature is that one dominant member of the chain controls the flow of goods in the way that is deemed most efficient and a means is devised for the rewards of this integrated control to be shared between the members of the chain. Over the years mathematical models have been developed to describe the behaviour of such integrated systems and to determine optimal control policies. Because such systems are complex much of this research has been concentrated on deterministic models in a slightly idealised settings. The hope and expectation is that these models provide some degree of qualitative insight into the behaviour of more complex real-world problems, which generally involve levels of uncertainty.

The basic model considered here consists of a single vendor who manufactures the final batch for a product and then transfers the stock to a single buyer as a number of shipments. The buyer has to satisfy from the stock a linearly decreasing and continuous demand process for a finite horizon. There are costs associated with batch setup, delivering a shipment, and holding stock at both the vendor and the buyer. The objective is to determine the shipment policy which minimises the total system cost. Goyal (1977) was probably one of the first papers to investigate the integrated single-supplier single-customer problem. Bannerjee (1986) considered the vendor manufacturing for stock at a finite rate and delivering the whole batch to the buyer as a single shipment—a 'lot for lot' model. Goyal (1988) demonstrated how lower cost policies generally result from allowing a production batch to be split and delivered as a number of shipments. Lu (1995) set out the optimal production and shipment policy when the shipment sizes are all equal. Goyal (1995) demonstrated how lower cost policies sometimes result when successive shipment sizes increase by a ratio which is equal to production rate divided by the demand rate. Hill (1999) derived the form of the optimal policy if shipment sizes may vary. This consists a number of
shipments which increase by the ratio used in Goyal (1995) followed by a number of equal-sized shipments.

Brosseau (1982) considered the optimal policy for a single level inventory problem when the demand rate is linearly decreasing and derived an analytical method for determining the optimal number of replenishments, the times of those replenishment and the corresponding optimum cost. After that several researchers investigated various approximate approaches to the problem. Recently, Goyal and Giri (2003) derived a simple rule by using a backward search method for determining the solution for the model with a linearly decreasing demand rate. However, most of the work on linearly decreasing demand is concerned with a single stocking point.

A common assumption that has been made is that the demand rate at the buyer is fixed over an infinite horizon. In this paper, demand up to time zero is constant, at rate \( a \). From time zero to time \( H \) the demand rate decreases linearly from \( a \) to zero. At time zero the vendor is about to make the final batch, at rate \( P \). The size of the batch \( Q \) will be exactly what is required to meet all remaining demand (up to time \( H \)). At time zero the buyer holds a quantity \( x \) in stock. The quantity \( x \) could be the amount at the beginning of the previous production cycle based on a fixed shipment size policy, or it could be based on a variable shipment size policy or it could just be an arbitrary amount of stock. However, there remains some demand for the product to be satisfied but this tails off (linearly). Just before the equipment for manufacturing the product is dismantled there is one final opportunity to make enough stock to meet all the remaining demand.

The problem is to find the optimal number of shipments and sizes of those shipments. Two possible policies would be for the shipments to have all the same size but not evenly spaced in time or to be evenly spaced in time but not equal in size. In this paper we consider the first case. For the globally optimal policy, the shipments would vary in both size and frequency.

In Section 2 we develop the mathematical formulation of the model. In Section 3 we look at a numerical examples and draw some conclusions in Section 4.

2. Mathematical formulation

In this section a general cost model will be developed.

2.1. Definitions and assumptions

To develop the model, the following terminology is used:

- The demand rate for the finished product at time \( t \) is \( f(t) = a(1 - t/H) \) for \( t \in (0, H) \). \( H \) is the time horizon.
- The finite production rate is \( P \) units per unit time.
- There is a fixed order/shipment cost of \( A_s \).
- There is an inventory carrying cost for the vendor of \( h_1 \) per unit per unit time for finished product.
- There is an inventory carrying cost for the buyer of \( h_2 \) per unit per unit time.
- \( n \) is the number of shipments.
- \( q_i \) is the size of the \( i \)th shipment in a final batch production run.
- \( x \) is the initial stock held at the buyer when the final production is about to start.
- \( C(n) \) is the total cost for the system for \( n \) shipments.

All the variables are assumed to be continuous rather than discrete in nature. No stock shortages at the buyer are allowed. All the parameters are deterministic and fixed over a finite planning horizon. It is assumed that \( P > a \). At time zero the vendor is about to make the final batch, to cover all remaining demand. Since we are only making one batch the production set up cost is irrelevant and can be ignored.

Let \( D \) be the total remaining demand, given by 

\[
D = \int_0^H f(t) \, dt = \frac{1}{2} aH.
\]

The initial amount of stock, \( x \), held at the buyer must be greater than 0 and if it is greater than or equal to \( D \) then we do not need to make a final batch at all. If we start with \( x \) units then we have to produce \( D - x \) units. If we want to dispatch this as \( n \) equal shipments then each shipment must be of size \( q = (D - x)/n \). The time \( t \) to produce the first shipment is \( q/P \). Let \( t_1 \) be the time for the buyer to use up the \( x \) units available, so that 

\[
\int_0^{t_1} f(t) \, dt = x.
\]

Solution for this \( t \leq H \) gives \( t_1 = H(1 - \sqrt{1 - 2x/aH}) \) if the initial stock at the buyer is less or equal to the total remaining demand \( x \leq \frac{1}{2} aH \). If \( t_1 \leq H \) then we have enough time to produce the first shipment and to dispatch as \( n \) equal shipments. The condition for this to hold is \( (D - x)/nP \leq t_1 \) or \( n \geq (D - x)/Pt_1 \). Fig. 1 gives the graphical representation for this policy when \( n = 4 \).

For a smaller \( x \), it is possible that \( n \) does not meet this condition. In this case, the first shipment size, \( Pt_1 \), is less than \( (D - x)/n \). For the second shipment, we move on to time \( t_1 \) and the new \( x \) becomes equal to \( Pt_1 \). At time \( t_1 \) we repeat this process except now \( n \) becomes \( n - 1 \). From this we re-compute the range of values for which the second and subsequent deliveries will give us equal shipment size. We repeat this process until we have allowed for a single remaining shipment.

2.2. Total time-weighted system stock

In Fig. 1 the stock levels, \( y_1 \) at time \( t \) in the interval \((tp, H)\) is \( \int_t^{l(t)} f(t) \, dt \) and \( y_2 \) in the interval \((0, t_1)\) is \( Pt + x - t_1 \)
\[ \int_0^t f(t) \, dt \] where \( t_p \) is the production up time. We also have
\[ R_p + x = \int_0^t f(t) \, dt \] and so \( t_p = (1/P)(D - x) \). The total time-weighted system stock, TSS, is
\[
\text{TSS} = \int_0^t y_2(t) \, dt + \int_{t_p}^H y_1(t) \, dt
\]
\[
= \frac{1}{2} \rho t^2_p + xt_p + \frac{1}{6} \alpha H^2 - \frac{1}{2} \alpha H t_p
\]
\[
= \frac{1}{6} \alpha H^2 - \frac{1}{2P} \left( \frac{1}{2} \alpha H - x \right)^2.
\] (1)

2.3. Total time-weighted buyer stock

The size of the ith shipment is \( q_i = \int_{t_i}^{t_{i+1}} f(t) \, dt \). Then, the time-weighted buyer stock from the ith shipment is
\[
\int_{t_i}^{t_{i+1}} \left[ a(t_{i+1} - t) - \frac{a}{2H} (t_{i+1}^2 - t^2) \right] \, dt
\]
\[
= at_{i+1}(t_{i+1} - t_i) - \frac{a}{2H} (t_{i+1}^2 - t_i^2)
- \frac{a}{2H} t_i^2 (t_{i+1} - t_i) + \frac{a}{6H} (t_{i+1}^3 - t_i^3).
\]

It follows that the total time-weighted buyer stock from \( n \) number of shipments, TBS, is
\[
\text{TBS} = a \sum_{i=1}^n (t_{i+1}^2 - t_i) - \frac{a}{2H} \sum_{i=1}^n (t_{i+1}^2 - t_i^2)
+ \frac{a}{6H} (H^3 - t_i^3).
\] (2)

For this policy, if \( \tau \leq t_1 \) then the remaining shipment times are given by
\[
t_{i+1} = H \left[ 1 - \sqrt{1 - \frac{2}{aH} (a_1 - \frac{1}{2H} a t_i^2 + \frac{1}{n} (D - x))} \right],
\] for \( i = 1, 2, \ldots, n \).

However, if \( \tau > t_1 \) or \( P_t \leq q \) then \( t_2 = H[1 - \sqrt{1 - (2/aH) (a_1 - (1/2H) a t_i^2 + P_t)}] \).

Next, if \( P(t_2 - t_1) \geq (1/(n-1))((D - x - P_t)) \) then the remaining shipment times are given by
\[
t_{i+1} = H \left[ 1 - \sqrt{1 - \frac{2}{aH} (a_1 - \frac{1}{2H} a t_i^2 + \frac{1}{n-1} (D - x - P_t))} \right],
\] for \( i = 2, 3, \ldots, n \).

Otherwise,
\[ t_1 = H[1 - \sqrt{1 - (2/aH) (a_2 - (1/2H) a t_1^2 + P(t_2 - t_1))}] \.
\]
We repeat this process until nth shipment. In order to satisfy the demand, we assume \( P(t_n - t_{n-1}) \geq (D - x - P_1 - \ldots - P(t_{n-1} - t_{n-2})) \).

Substitute all values of \( t \) in Eq. (2) gives the total TBS for this policy. The total cost for this policy, \( C \), is
\[ C(n) = n A_2 + h_1 TSS + (h_2 - h_1) \text{TBS}. \] (3)

Using Eq. (3), we can determine the minimum total cost of the system. Unfortunately, we cannot guarantee that the total cost function is convex in \( n \) and remains as a conjectures. However, since the minimum cost for \( n \) shipments is at least \( n A_2 + h_1 \text{TSS} \) we may stop our search through \( n \) as soon as \( n \) is greater than \( (\text{minimum cost found so far} - h_1 \text{TSS})/A_2 \).

3. Numerical illustration

To demonstrate the effectiveness of these models, we present some numerical examples. In this example, \( f(t) = 100(1 - t/H) \) for \( t \in (0, 5) \). Other parameters are:
\[ A_2 = 25, \ h_1 = 4, \ h_2 = 5, \ P = 1000, \] and \( x = 15 \).

For this example we have enough time to produce the first and the subsequent batches as \( n \) equal shipments for all cases since \( t_1 (= 0.152) \) is greater than \( \tau \) for all values of \( n \geq 2 \). The value of \( \tau \) for \( n = 1 \) is 0.235. Table 1 gives the minimum cost for the model together with the shipment size, cost for the vendor and buyer. The cost for the vendor increases as \( n \) increases while the cost for the buyer decreases. The minimum cost for the model is 1774.26 with the shipment size 58.75. The delivering shipment times for the optimal policy are at 0.152, 0.802, 1.572 and 2.576, respectively, with the value of \( \tau \) is 0.058. We stop our search when \( n = 9 \) since \( (\text{minimum cost found so far} - h_1 \text{TSS})/A_2 = 8.72 \).

Table 2 gives the minimum cost for the system, vendor and buyer for each \( n \) when \( x = 5 \). We have to begin with \( n > 1 \) since \( x + P_t \) is less than the total remaining demand \( (1/2aH) \). However, the first shipment size is less than the subsequent sizes when \( n = 2, 3, 4 \) because the value of \( \tau \) for these cases is greater than \( t_1 \). For example, the shipment sizes when \( n = 3 \) are 50.25, 97.35 and 97.35 with the minimum cost 1795.71. The optimum cost for this problem is 1772.61 when \( n = 5 \). All the shipment sizes for this optimal policy are equal since \( \tau (= 0.049) \) is less than \( t_1 (= 0.050) \). Similarly, we stop our search when \( n = 10 \) since \( (\text{minimum cost found so far} - h_1 \text{TSS})/A_2 = 9.04 \).

Table 1
The optimal cost when \( x = 15 \)

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<tr>
<th>( n )</th>
<th>Cost (vendor)</th>
<th>Cost (buyer)</th>
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<tbody>
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<td>2</td>
<td>1821.40</td>
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<td>3</td>
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<td>8</td>
<td>1819.98</td>
<td>29.38</td>
</tr>
<tr>
<td>9</td>
<td>1838.61</td>
<td>26.11</td>
</tr>
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</table>

Table 2
The optimal cost when \( x = 5 \)

<table>
<thead>
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<th>( n )</th>
<th>Cost (vendor)</th>
<th>Cost (buyer)</th>
</tr>
</thead>
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<td>1850.04</td>
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</table>
4. Conclusion

In this paper we have extended the previous work to the case where demand rate at the buyer is linearly decreasing with time to zero at the end of planning horizon. If the demand rate decreases at a gradual rate (or when $H$ is relatively large), it would be more economical to produce two or more batches for satisfying the remaining demand. However, in this paper we only consider the case where we have one final batch with equal shipment lots size to the buyer.

References


