A production–repair inventory model with time-varying demand and multiple setups

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ARTICLE INFO

Article history:
Received 12 June 2013
Accepted 30 December 2013
Available online 8 January 2014

Keywords:
Inventory
Production
Time-varying demand
Reuse

ABSTRACT

In this paper, we proposed a model for an inventory system that satisfies a continuous time-varying demand for a finished product over a known and finite planning horizon by supplying both new and repaired items. New items are fabricated from a single type of raw material procured from external suppliers, while used items are collected from the customers and then repaired to a condition that is as-good-as-new. During each time interval, new items and used items are produced from multiple production and repair runs. The problem is to determine a joint policy for raw materials procurement, new items fabrication, and used items repair such that the total relevant cost of the model is minimized. We also proposed a numerical solution procedure and we tested the model with some numerical examples and a simple sensitivity analysis.

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1. Introduction

Globally, urbanization and competitive markets drive the rise in consumption of manufactured products, resulting in faster and faster depletion of natural resources and accumulation of solid waste. The composition of solid waste is more complex than before due to the diffusion of plastics and electronic consumer products. The environmental problems posed by these phenomena led governments around the world to legislate laws that require some manufacturers to recycle used items that have reached the end of their useful lives. Thus it is desirable for these manufacturers to adopt recycling policies that economize their operating costs.

In the literature of inventory management, the study of inventory models that incorporate recycling started with authors modifying the economic order quantity (EOQ) model by Harris (1913), as the simple mathematics of the EOQ model makes it an appropriate base. Schrady (1967) proposed an EOQ model with instantaneous order and repair rates. His model treated the serviceable and recoverable inventories as interdependent parts of a single system, and jointly determined the optimal order and repair quantities for (1, R) policies – policies that alternate one order batch with R repair batches – using expressions whose derivations are similar to the classical EOQ formula. The work of Schrady (1967) was extended for the case of finite repair rate (Nahmias and Rivera, 1979), and for the case of multiple products (Mabini et al., 1992). Koh et al. (2002) also extended the work of Schrady (1967) for the case of finite repair rate, but they allowed the repair rate to be less than or equal to the demand rate, and they considered (P, 1) policies – policies that alternate one repair batch with P production batches – as well. They found the optimal setup numbers using exhaustive search procedures. Later, Teunter (2004) generalized earlier works by considering the case of finite production rate, but he used a semi-heuristic procedure to ensure the discreteness of P and R. In follow-up works, Konstantaras and Papachristos (2008a) proposed exact solution procedures for the model by Koh et al. (2002), and Konstantaras and Papachristos (2008b) and Wee and Widyadana (2010) proposed exact solution procedures for the model by Teunter (2004).

The above works assumed that all returned items are recovered. Richter (1996a), Richter (1996b) studied an EOQ waste disposal model, in which some returned items are scrapped. He assumed instantaneous production and repair rates, and multiple production and repair setups during each cycle. Richter (1997) extended the cost analysis of his earlier works and obtained an extremal result: the pure strategy of total repair or the pure strategy of total disposal dominates any mixed strategy of repair and disposal. Richter and Dobos (1999) and Dobos and Richter (2000) extended the works of Richter (1996a), Richter (1996b), Richter (1997) by considering part of the problem as an integer programming problem to secure the discreteness of the setup numbers. In a similar work, Teunter (2001) examined an EOQ waste disposal model with different holding costs for recovered and manufactured items. Dobos and Richter (2003) relaxed the

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http://dx.doi.org/10.1016/j.ijpe.2013.12.034
assumption of instantaneous production and repair rates by considering a finite rate waste disposal model with a single production cycle and a single repair cycle per time interval. Dobos and Richter (2004) generalized their earlier work by considering multiple production and repair cycles per time interval. The results of these works reiterated the dominance of pure strategies vis-a-vis mixed strategies. El Saadany and Jaber (2008) extended the works of Richter (1996a), Richter (1996b) by proposing a modified model that does not ignore the first time interval when no repairing occurs. Their unit time cost is lower than Richter’s, because Richter took unnecessary residual inventory into account, resulting in the holding costs being overestimated.

Dobos and Richter (2004) raised the question about the quality of returned items and the party to control this quality. This question was addressed when Dobos and Richter (2006) found that the decision to control the quality of returned items inhouse or to outsource it depends on whether the non-EOQ costs are taken into account or not. El Saadany and Jaber (2010) considered the case of a variable return rate that depends on the purchasing price and the quality level of the returned items, and they showed that pure strategies are no longer optimal against mixed strategies.

The above works assumed that recovered items are of a quality that is as-good-as-new. Jaber and El Saadany (2009) addressed this limitation by assuming that the demand for remanufactured items is different from the demand for newly manufactured ones, i.e., there is a quality difference between remanufactured items and new items. Konstantaras et al. (2010b) assumed that returned items are either refurbished, sold at a secondary market, and never collected again, or remanufactured to a condition that is as-good-as-new.

Most of the above works do not account for shortages, which can be economical in some situations. Konstantaras and Papachristos (2006) obtained an optimal production and recovery policy analytically for an EOQ model with complete backorders. Konstantaras and Skouri (2010a) obtained sufficient conditions for the optimal policy of a model with complete backorders and variable setup numbers. Hasanov et al. (2012) extended the work of Jaber and El Saadany (2009) for the case of pure and partial backordering.

Other authors who wrote relevant works include Choi et al. (2007), who extended the work of Koh et al. (2002) for the case of (P, R) policies with the sequence of P orders and R repairs itself as a decision variable; Jaber and Rosen (2008), who extended the works of Richter (1996a), Richter (1996b) by proposing entropy costs to address the difficulty of estimating the EOQ cost parameters; Jaber and El Saadany (2011), who considered learning effects; El Saadany and Jaber (2011), who considered the case of serviceable items being manufactured from a mixture of new or remanufactured subassemblies; and El Saadany et al. (2013), who extended the works of Richter (1997) and Teunter (2001) for the case of limited remanufacturability.

Besides stationary demand, some authors have studied reuse models with time-varying demand functions. Omar and Yeo (2009) proposed an integrated production–repair model that held three types of items in stock: used items, serviceable items, and raw materials. The producer serves a continuous time-varying demand over a finite planning horizon by producing new items from raw materials as well as by repairing used items. Each time interval is divided into production and repair periods, each with multiple setups. However, they assumed that used items are not collected during the repair period. They described a numerical solution procedure to find the interval times and setup numbers that minimize the total relevant cost. Alamri (2011) presented a global optimal solution to a general reverse logistics model with time-varying rates for demand, return, production, repair, and deterioration. They considered one remanufacturing setup and one production setup per time interval, over an infinite planning horizon.

In this paper, we extend the work of Omar and Yeo (2009) by relaxing the assumption that used items are not collected during the repair period. A general framework of the material flow through the relevant inventories is given in Fig. 1. The remainder of this paper is organized as follows. In Section 2, the mathematical formulation of the proposed model is developed; in Section 3, a numerical solution procedure is proposed; in Section 4, the model is tested with some numerical examples; in Section 5, a simple sensitivity analysis is performed; and finally, in Section 6, the paper is concluded.

2. Model formulation

In this paper, we make the following assumptions:

1. A single item inventory system that operates over a known and finite planning horizon of length H units of time.

2. The demand rate D(t) is a deterministic and known function of time t, with D(t) > 0 for 0 ≤ t ≤ H.

3. The production rate P and repair rate R are finite and constant, with P > D(t) and R > D(t) for 0 ≤ t ≤ H.

4. The return rate C(t) is linearly proportional to the demand rate, i.e., C(t) = θD(t), with 0 < θ < 1.

5. All used items are repaired. Repaired items are as good as new.

6. Only one type of raw material is required to fabricate new items. After an order is placed, the raw materials are immediately replenished.

7. Each time interval has ν (ν = 1, 2, …) production runs and raw materials orders, and w (w = 1, 2, …) repair runs, with one raw materials order per production run.

8. Used items are not collected during the repair runs of the final time interval.

9. Shortages are not allowed during the planning horizon.

10. The following cost parameters are considered:

   (a) k_p, the setup cost of each production run (cost/setup).

   (b) k_r, the setup cost of each repair run (cost/setup).

   (c) k_m, the ordering cost of the raw materials (cost/order).

   (d) h_p, the inventory holding cost of the serviceable items (cost/unit/time).

   (e) h_s, the inventory holding cost of the used items (cost/unit/time).

   (f) h_m, the inventory holding cost of the raw materials (cost/unit/time).

   (g) s_p, the unit production cost of the new items (cost/unit).

   (h) s_r, the unit repair cost of the used items (cost/unit).

Note that the unit production and repair costs may be omitted from the total inventory cost as the return portion is fixed, but
the inclusion of these costs enables us to compare the model with other models that take these costs into account.

In addition, we use the following notations:

1. \( n \), the number of time intervals during the planning horizon \((n = 1, 2, \ldots)\).
2. \( m \), the amount of raw materials that are required to produce one new item.
3. \( t_i \), the total elapsed time up to the start of the \((i+1)\)th time interval’s \((i = 0, 1, \ldots, n - 1)\) first production run.
4. \( \beta_i \), the total elapsed time up to the start of the \((i+1)\)th time interval’s \((i = 0, 1, \ldots, n - 1)\) first repair run.
5. \( p_{ij} \), the total elapsed time up to the start of the \(j\)th production run \((j = 1, 2, \ldots, v)\) during the \((i+1)\)th time interval.
6. \( \alpha_{ij} \), the total elapsed time up to the end of the \(j\)th production run during the \((i+1)\)th time interval.
7. \( r_{ik} \), the total elapsed time up to the start of the \(k\)th repair run \((k = 1, 2, \ldots, w)\) during the \((i+1)\)th time interval.
8. \( \gamma_{ik} \), the total elapsed time up to the end of the \(k\)th repair run during the \((i+1)\)th time interval.
9. \( u_{i} \), the inventory level of the used items at the start of the \((i+1)\)th time interval.

Fig. 2 shows the inventory movement of the used items, the serviceable items, and the raw materials during the \((i+1)\)th time interval, when demand is increasing over time and \(v = w = 2\). The \(i \leq t \leq \beta_i\) period is called the production period and the \(\beta_i \leq t \leq t_{i+1}\) period is called the repair period. The figure also shows the inventory movement during the repair period of the final time interval, when the collection of used items has stopped.

The total relevant cost over the planning horizon when there are \(n\) time intervals, \(v\) production runs and raw materials orders, and \(w\) repair runs is given by

\[
TRC = n(vk_p + wk_t + vK_m) + H_p + H_t + H_m + S_p + S_t, \tag{1}
\]

where \(H_p, H_t,\) and \(H_m\) are the total inventory holding costs of the serviceable items, the used items, and the raw materials, respectively; \(S_p\) is the total production cost; and \(S_t\) is the total repair cost.

The production period of each time interval can be divided into a series of uptime and downtime periods as follows:

\[
t_i < a_{i,1} < p_{i,2} < a_{i,2} < p_{i,3} < \cdots < p_{i,v} < a_{i,v} < \beta_i, \quad i = 0, 1, \ldots, n - 1. \tag{2}
\]

Similarly, the repair period of each time interval can also be divided into a series of uptime and downtime periods as follows:

\[
\beta_i < \gamma_{i,1} < r_{i,2} < \gamma_{i,2} < r_{i,3} < \cdots < r_{i,w} < \gamma_{i,w} < t_{i+1}. \quad i = 0, 1, \ldots, n - 1. \tag{3}
\]

Since the amount of serviceable items produced (or repaired) after an uptime period is equal to the demand during both the uptime and downtime periods, the inventory level of the serviceable items with respect to time, \(l_p(t)\), is given by

\[
l_p(t) = \begin{cases}
\left(R(t - p_{ij}) - f_{j}^{\alpha_{ij}} D(z) dz, & \alpha_{ij} \leq t \leq \beta_{ij}, \quad j = 1, 2, \ldots, v, \\
p_{ij} - \left(R(t - r_{ik}) - f_{i}^{\gamma_{ik}} D(z) dz, & \gamma_{ik} \leq t \leq \beta_{ik}, \quad k = 1, 2, \ldots, w, \right. \\
& \beta_{ik} \leq t \leq r_{ik+1}, \quad k = 1, 2, \ldots, w, \\
& \beta_{ik} \leq t \leq r_{ik+1}, \end{cases}
\]

where \(p_{i} = t_i + (j - 1)X_t, \quad j = 1, 2, \ldots, v, \)

\[
r_{ik} = \beta_{i} + (k - 1)Y_i, \quad k = 1, 2, \ldots, w, \tag{4}
\]

In a time-varying demand environment, the uptime and downtime periods within each time interval may have varying lengths. However, for simplicity, we fix the production times and the repair times to occur at constant intervals within the production period and the repair period, respectively. This gives us

\[
p_{ij} = t_i + (j - 1)X_i, \quad j = 1, 2, \ldots, v, \tag{5}
\]

\[
r_{ik} = \beta_{i} + (k - 1)Y_i, \quad k = 1, 2, \ldots, w, \tag{6}
\]

where \(X_i = (\beta_{i} - t_i)/v\) and \(Y_i = (t_{i+1} - \beta_{i})/w\).

The total inventory holding cost of the serviceable items throughout the planning horizon, \(H_p\), is given by

\[
H_p = h_p \sum_{i=0}^{n} \left\{ \sum_{j=1}^{v} \left[ \int_{p_{ij}}^{\alpha_{ij}} l_p(t) dt + \sum_{k=1}^{w} \left[ \int_{r_{ik}}^{\gamma_{ik}} l_p(t) dt \right] \right] \right\}. \tag{7}
\]

Let \(g(t)\) be an antiderivative of \(D(t)\). We obtain

\[
\int_{a_{ij}}^{p_{ij}} l_p(t) dt = \frac{P}{2}(\alpha_{ij} - p_{ij})^2 + g(p_{ij})(\alpha_{ij} - p_{ij}) - \int_{p_{ij}}^{p_{ij+1}} g(t) dt, \tag{10}
\]

\[
\int_{p_{ij+1}}^{p_{ij+1}} l_p(t) dt = g(p_{ij+1})(p_{ij+1} - \alpha_{ij}) - \int_{p_{ij+1}}^{p_{ij+1}} g(t) dt, \tag{11}
\]

\[
\int_{r_{ik}}^{\gamma_{ik}} l_p(t) dt = \frac{R}{2}(\gamma_{ik} - r_{ik})^2 + g(r_{ik})(\gamma_{ik} - r_{ik}) - \int_{r_{ik}}^{r_{ik+1}} g(t) dt, \tag{12}
\]

\[
\int_{r_{ik+1}}^{r_{ik+1}} l_p(t) dt = g(r_{ik+1})(r_{ik+1} - \gamma_{ik}) - \int_{r_{ik+1}}^{r_{ik+1}} g(t) dt. \tag{13}
\]

Using Eqs. (5)–(8), we obtain

\[
H_p = h_p \sum_{i=0}^{n} \sum_{j=1}^{v} \left[ X_i g(t_i + jX_i) - \frac{1}{2P} \left( \int_{t_i + jX_i}^{t_{i+1} + (j+1)X_i} \frac{D(z) dz}{D(z)} \right)^2 \right].
\]
+ \sum_{k=1}^{w} Y_i g(\beta_i + kY_i) - \frac{1}{2R} \left( \int_{\beta_i \rightarrow (k-1)Y_i} T(t) \, dt \right)^2 - \int_{t_i}^{t_{i+1}} g(t) \, dt \right).  \\
(14)
\]

Since used items are collected continuously, and are consumed only during the repair uptime periods, the inventory level of the used items with respect to time, \( I_i(t) \), is given by
\[
I_i(t) = \begin{cases} 
  u_i + \int_{\beta_i}^{t_i} D(z) \, dz, & t_i \leq t \leq \beta_i, \\
  u_i + \int_{\beta_i}^{t_i} D(z) \, dz - R(t - r_{i,k}), & r_{i,k} \leq t \leq \beta_i, \\
  u_i + \int_{\beta_i}^{t_i} D(z) \, dz - \sum_{m=1}^{k-1} R(\gamma_i,m - r_{i,m}) - R(t - r_{i,k}), & r_{i,k} \leq t \leq \gamma_i, \\
  u_i + \int_{\beta_i}^{t_i} D(z) \, dz - \sum_{m=1}^{k-1} R(\gamma_i,m - r_{i,m}), & \gamma_i \leq t \leq \beta_i + 1, \\
  \cdots 
\end{cases}
\]
\[
(15)
\]

Note that the last three expressions do not apply for the case of \( i = n - 1 \) (the final time interval) because used items are not collected during the repair period of this time interval. Moreover, the inventory level of the used items at the end of each time interval (except the final one), \( u_i, i \), is given by
\[
u_{i+1} = u_i + \theta \int_{\beta_i}^{t_{i+1}} D(t) \, dt - \int_{t_i}^{t_{i+1}} D(t) \, dt.
\]
\[
(16)
\]

Observe that \( u_{i+1} \) is a recursive function of \( u_i \) with \( u_0 \) being a known constant.

The total inventory holding cost of the used items throughout the planning horizon, \( H_t \), is given by
\[
H_t = H_{t,n} = \sum_{i=1}^{n-2} \left\{ \int_{\beta_i}^{t_i} I_i(t) \, dt + \sum_{k=1}^{w} \int_{\beta_i}^{t_{i+1}} I_i(t) \, dt \right\} + H_{t,n},
\]
\[
(17)
\]

where \( H_{t,n} \) is the inventory holding cost of the used items during the final (n-th) time interval, and
\[
\int_{t_i}^{t_{i+1}} I_i(t) \, dt = [u_i - \theta g(t_i)](t_{i+1} - t_i) + \theta \int_{t_i}^{t_{i+1}} g(t) \, dt,
\]
\[
(18)
\]

Thus the inventory holding cost of the used items during the final time interval is given by
\[
H_{t,n} = h_n \left\{ \int_{t_{n-1}}^{t_{n}} I_i(t) \, dt + \sum_{k=1}^{w} A_{n-1,k} + \sum_{k=1}^{w-1} B_{n-1,k} \right\},
\]
\[
(23)
\]

Using Eqs. (8), (18), (21), and (22), we obtain
\[
H_{t,n} = h_n \left\{ \left[ u_{n-1} - \theta g(t_{n-1}) \right] \left[ \beta_{n-1} - t_{n-1} \right] + \theta \int_{t_{n-1}}^{t_{n}} g(t) \, dt \right\}
\]
\[
+ \sum_{k=1}^{w} \left\{ \frac{1}{2R} \left( \int_{\beta_{n-1} + (k-1)Y_i}^{\beta_{n-1} + kY_i} D(t) \, dt \right)^2 - \int_{t_{n-1}}^{t_{n}} Y_i g(t) \, dt \right\}
\]
\[
(24)
\]

Hence using Eqs. (6), (8), (17)–(20), and (24), we obtain
\[
H_t = h_t \sum_{i=1}^{n-2} \left\{ [u_i - \theta g(t_i)](t_{i+1} - t_i) + \theta \int_{t_i}^{t_{i+1}} g(t) \, dt \right\}
\]
\[
+ \sum_{k=1}^{w} \left\{ \frac{1}{2R} \left( \int_{\beta_i + (k-1)Y_i}^{\beta_i + kY_i} D(t) \, dt \right)^2 - \int_{t_i}^{t_{i+1}} Y_i g(t) \, dt \right\}
\]
\[
(25)
\]

Since there is only one order of raw materials per production run, and there are \( v \) production runs per time interval, the time-weighted inventory holding of the raw materials during the \( (i+1) \) th time interval is given by the sum of the areas of \( v \) right triangles, where \( C_{ij} \) is the area of the corresponding triangle, as shown in Fig. 2. Using Eqs. (5) and (7), the total inventory holding cost of the raw materials throughout the planning horizon, \( H_m \), is given by
\[
H_m = h_m \sum_{i=1}^{n} \sum_{j=1}^{v} C_{ij} = h_m h \sum_{i=0}^{n} \sum_{j=0}^{v} \left\{ \int_{t_i}^{t_{i+1}} D(t) \, dt \right\}^2.
\]
\[
(26)
\]

Since used items are collected continuously from \( t=0 \) to \( t=\beta_{i-1} \) and all units are repaired, the total repair cost is given by
\[
S_r = \sum_{i=1}^{n} \left( u_0 + \theta \int_{0}^{\beta_{i-1}} D(t) \, dt \right).
\]
\[
(27)
\]

On the other hand, since the demand throughout the planning horizon is served by both new and repaired items, the total production cost is given by
\[
S_p = \sum_{i=1}^{n} \left( \int_{0}^{\beta_{i-1}} D(t) \, dt - u_0 - \theta \int_{0}^{\beta_{i-1}} D(t) \, dt \right).
\]
\[
(28)
\]

Moreover, the collection policy is fixed such that for each time interval, when the first repair run commences, the amount of used items available is equal to the amount of serviceable items demanded during the repair period, i.e.
\[
u_i + \int_{t_i}^{t_{i+1}} \theta D(t) \, dt = \int_{t_i}^{t_{i+1}} D(t) \, dt.
\]
\[
(29)
\]

Thus \( \beta_i \) is a function of \( (t_i, t_{i+1}, u_i) \) and can be found by solving the equation above.

Finally, after some algebraic manipulation, the total relevant cost can be written as
\[
TRC(n, v, w, t) = \pi (\psi_{rp} + \psi_{tr} + \psi_{km})
\]
observations of such behaviors in all experiments of the solution procedure.

As mentioned, for fixed \( n \) and \( v \), an improvement (decrease) in \( TRC(n, v, W, \mathbf{t}^*) \) is expected to be observed as \( w \) increases until \( w=W \). Beyond this, increasing \( w \) should increase \( TRC(n, v, W, \mathbf{t}^*) \). Thus there is a positive integer \( k_2 \) such that

\[
TRC(n, v, W, \mathbf{t}^*) > n[v(k_2+k_0)+(W+k_2)k_1] > TRC(n, v, W, \mathbf{t}^*)
\]

for any \( w \geq W+k_2 \). Hence \((W+k_2)\) is an upper bound for \( w \) until which the evaluation of \( TRC(n, v, W, \mathbf{t}^*) \) should be conducted to test the validity of \( TRC(n, v, W, \mathbf{t}^*) < TRC(n, v, W, \mathbf{t}^*) \) for any \( w > W \). Similarly, there are positive integers \( k_1 \) and \( k_0 \) such that

\[
TRC(n, v, W, \mathbf{t}^*) > n[v(k_1+k_0)+(W+k_1)k_0] > TRC(n, V, W, \mathbf{t}^*)
\]

(32)

\[
TRC(n, V, W, \mathbf{t}^*) > (N+k_0)(V(k_2+k_0)+Wk_2] > TRC(N, V, W, \mathbf{t}^*)
\]

(33)

for any \( v \geq V+k_1 \) and \( n \geq N+k_0 \), respectively.

The computer algorithm of the solution procedure is outlined below. Note that when \( n=1, \mathbf{t}^* \) is given by \( t_0=0 \). On the other hand, when \( n>1, \mathbf{t}^* \) is found by improving the initial solution \( \mathbf{t}=(0, H/n, 2H/n, \ldots, H(n-1)/n) \).

1. Set \( n=1, v=1, \) and \( w=1 \).

2. Do

\( (a) \) compute the best \( TRC(n, v, w, \mathbf{t}^*) \),

\( (b) \) increment \( w \) by 1,

while \( TRC(n, v, w, \mathbf{t}^*) < TRC(n, v, w, \mathbf{t}^*) \).

3. Record \( W=w-1 \). Verify that \( TRC(n, v, w, \mathbf{t}^*) \) for \( w \) from \( W+1 \) to \( W+k_2 \) are all greater than \( TRC(n, v, W, \mathbf{t}^*) \), with \( k_2 \) satisfying Inequality (31).

4. Do

\( (a) \) increment \( v \) by 1. Set \( w=1, \)

\( (b) \) step 2 to 3,

while \( TRC(n, v, W, \mathbf{t}^*) < TRC(n, v, W, \mathbf{t}^*) \).

5. Record \( V=v-1 \). Verify that \( TRC(n, v, W, \mathbf{t}^*) \) for \( v \) from \( V+1 \) to \( V+k_1 \) are all greater than \( TRC(n, v, W, \mathbf{t}^*) \), with \( k_1 \) satisfying Inequality (32).

6. Do

\( (a) \) increment \( n \) by 1. Set \( v=0, \)

\( (b) \) step 4 to 5,

while \( TRC(n, V, W, \mathbf{t}^*) < TRC(n, V, W, \mathbf{t}^*) \).

7. Record \( N=n-1 \). Verify that \( TRC(n, V, W, \mathbf{t}^*) \) for \( n \) from \( N+1 \) to \( N+k_0 \) are all greater than \( TRC(n, V, W, \mathbf{t}^*) \), with \( k_0 \) satisfying Inequality (33).

8. Return \((N, V, W, \mathbf{t}^*)\) as the best solution.

4. Numerical examples

**Example 1.** We illustrate the use of the proposed solution procedure by solving a specific problem. Suppose that the demand function is a function of the form \( D(t)=at+b \), where \( a>0 \) and \( b \neq 0 \). Also, suppose that an initial solution \( \mathbf{t}=(H/n), \) where \( i=0, 1, \ldots, n-1 \), is used to start the solution procedure. From Eqs. (29) and (16), respectively, we have

\[
\beta_i = \frac{b}{a} \left[ \frac{b^2}{a^2} + \frac{2a}{a^2} + \frac{a^2}{a^2} \right] \frac{1}{2} \left( t_{i+1} - t_i \right)^2 + \frac{b(t_{i+1} + t_i)}{a} \frac{1}{2} \left( t_{i+1} + t_i \right)
\]

(34)
\[ u_{i+1} = u_i + \frac{a}{b} \left( t_{i+1} - t_i \right) + b \theta (t_{i+1} - t_i) - \frac{a}{b} \beta_i \left( t_{i+1} - t_i \right), \quad i = 0, 1, ..., n - 2. \]  
(35)

Note that in this case, \( \beta_i \) is a unique root of Eq. (29) because the negative root is rejected. For a fixed \( n \), since \( u_0, t_0 \), and \( t_1 \) are known, \( \beta_i \) and \( u_2 \) can be obtained using Eqs. (34) and (35), respectively. In general, observe that once \( \beta_i \) and \( u_i \) are known, \( \beta_i \) and \( u_{i+1} \) are obtained as well.

The parameter values for this example are listed in Table 1. The best total relevant cost, \( TRC(N, V, W, \mathbf{t}^*) = 8437 \), with \( N = 4, V = 1 \), and \( W = 3 \). The best values of \( \mathbf{t}, \mathbf{B}, \mathbf{w} \) are tabulated in Table 2. The values of \( TRC(n, v, w, \mathbf{t}^*) \) for varying \( n, v, \) and \( w \) are tabulated in Table 3.

**Example 2.** This example uses the same parameter values as Example 1, with the exception that \( k_p = 5 \) and \( k_r = 500 \) this time. The best total relevant cost, \( TRC(N, V, W, \mathbf{t}^*) = 8061 \), with \( N = 3, V = 5 \), and \( W = 1 \). The best values of \( \mathbf{t}, \mathbf{B}, \) and \( \mathbf{u} \) are tabulated in Table 4. The system adopts a \( (P, 1) \) policy when \( k_p < k_r \), whereas it adopts a \( (1, R) \) policy when \( k_p > k_r \) (Example 1). The system has never adopted a \( (P, R) \) policy in any of the numerical experiments that we performed.

### Table 1
Input parameter values for Example 1.

<table>
<thead>
<tr>
<th>( k_p )</th>
<th>( k_r )</th>
<th>( k_m )</th>
<th>( h_p )</th>
<th>( h_i )</th>
<th>( h_m )</th>
<th>( s_p )</th>
<th>( s_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>5</td>
<td>10</td>
<td>30</td>
<td>15</td>
<td>5</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>( P )</td>
<td>( R )</td>
<td>( \theta )</td>
<td>( a )</td>
<td>( b )</td>
<td>( m )</td>
<td>( u_0 )</td>
<td>( H )</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>0.7</td>
<td>15</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

### Table 2
Best values of \( \mathbf{t}, \mathbf{B}, \) and \( \mathbf{u} \) for Example 1.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
<th>( t_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>1.6478</td>
<td>2.9235</td>
<td>4.0142</td>
<td>5</td>
</tr>
<tr>
<td>( \beta_i )</td>
<td>1.1914</td>
<td>2.3459</td>
<td>3.4166</td>
<td>4.4115</td>
</tr>
<tr>
<td>( u_i )</td>
<td>0</td>
<td>8.7194</td>
<td>18.4027</td>
<td>25.8216</td>
</tr>
</tbody>
</table>

### Table 3
\( TRC(n, v, W, \mathbf{t}^*) \) for varying \( n, v, \) and \( w \) for Example 1.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( v )</th>
<th>( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>1</td>
<td>15,354</td>
</tr>
<tr>
<td>2</td>
<td>1,1632</td>
<td>11,489</td>
</tr>
<tr>
<td>3*</td>
<td>11,046</td>
<td>10,903</td>
</tr>
<tr>
<td>4</td>
<td>11,059</td>
<td>10,917</td>
</tr>
<tr>
<td>2</td>
<td>10,003</td>
<td>9878</td>
</tr>
<tr>
<td>2*</td>
<td>9802</td>
<td>9675</td>
</tr>
<tr>
<td>3</td>
<td>10,458</td>
<td>10,330</td>
</tr>
<tr>
<td>4</td>
<td>11,302</td>
<td>11,174</td>
</tr>
<tr>
<td>3</td>
<td>1*</td>
<td>8869</td>
</tr>
<tr>
<td>2</td>
<td>9772</td>
<td>9665</td>
</tr>
<tr>
<td>3</td>
<td>1,1101</td>
<td>10,992</td>
</tr>
<tr>
<td>4</td>
<td>12,531</td>
<td>12,421</td>
</tr>
<tr>
<td>4*</td>
<td>1*</td>
<td>8544</td>
</tr>
<tr>
<td>2</td>
<td>10,182</td>
<td>10,090</td>
</tr>
<tr>
<td>3</td>
<td>12,088</td>
<td>11,995</td>
</tr>
<tr>
<td>4</td>
<td>14,060</td>
<td>13,966</td>
</tr>
<tr>
<td>5</td>
<td>1*</td>
<td>8529</td>
</tr>
<tr>
<td>2</td>
<td>10,797</td>
<td>10,713</td>
</tr>
<tr>
<td>3</td>
<td>13,242</td>
<td>13,164</td>
</tr>
<tr>
<td>4</td>
<td>15,742</td>
<td>15,663</td>
</tr>
</tbody>
</table>

* \( t^* \) is the best \( t \) found by the solution procedure for the given \( n, v, w \).
results are presented graphically in Figs. 3–8. The parameter values of $k_p$, $k_r$, $h_p$, and $h_r$ are changed over the range of $[1/4, 1/2, 1, 2, 4]$ of their original values, while the parameter values of $k_m$ and $h_m$ are changed over the range of $[1, 5, 10, 20, 50]$ of their original values. Note that the best production setup number $V$ is always found to be equal to 1.

Fig. 3 shows that as $k_p$ increases, $N$ decreases while $W$ increases. This is because as $k_p$ increases, the system favors smaller interval numbers. However, at the same time, it compensates the consequent increase in stock holding by reducing the stock on hand during the repair period. This is achieved by increasing the number of repair runs (setups).

Fig. 4 shows that as $k_r$ increases, $W$ decreases while $N$ remains unchanged. This is because as $k_r$ increases, the system favors smaller repair setup numbers. The reason $N$ remains unchanged is because the change in $k_r$ is not large enough (relatively) to have a noticeable effect on $N$ (the ratio of the base value of $k_r$ to $k_p$ is 1:100). In other words, the system finds no cost improvement by increasing $N$ to compensate for the increase in stock holding as the number of repair runs decreases.

Fig. 5 shows that as $k_m$ increases, $N$ decreases while $W$ increases. The change in $W$ is tied to the change in $N$, as the system compensates for the increase in stock holding from smaller
interval numbers by running more repairs. However, the changes in \( N \) and \( W \) are small, despite \( k_m \) increasing its value by 50 times over the test range, because \( k_m \) is relatively small compared to \( h_p \). (The ratio of their base value is 1:50).

Fig. 6 shows that as \( h_p \) increases, \( W \) increases while \( N \) remains unchanged. The former is because as \( h_p \) increases, the system reduces stock holding of serviceable items by running more repairs. However, instead of reducing stock holding by running more productions per time interval, or by running more time intervals, the system surprisingly maintains the best production setup number and the best interval number. We note that \( N \) would increase with \( h_p \), had the change in \( h_p \) been relatively larger.

Fig. 7 shows that as \( h_r \) increases, \( N \) increases while \( W \) decreases. This is because as \( h_r \) increases, the system favors larger interval numbers to reduce the stock holding of used items. But at the same time, it favors smaller repair setup numbers, because running multiple repair setups per time interval increases the stock holding of used items, which is traded off against the reduction in the stock holding of serviceable items.

Fig. 8 shows that as \( h_m \) increases, \( N \) increases while \( W \) remains unchanged. This is because as \( h_m \) increases, the system favors larger interval numbers to reduce the stock holding of raw materials. However, the change in \( N \) is not as large as it is when \( h_p \) or \( h_r \) are increasing, because raw materials inventory comprises a small portion of system inventory, and hence its contribution to the total holding costs is relatively small. We note that as the raw materials requirement for the unit production of the finished product increases, the rate at which \( N \) increases would increase as well.

6. Conclusion

In this paper, we formulated an inventory model for the integrated production of new items and repair of used items. The planning horizon is known and finite in length. The demand rate is a deterministic function of time, while the production and repair rates take constant values. We proposed a numerical solution procedure, and we gave numerical examples to illustrate this procedure. The numerical results also illustrated how changes in parameter values affect the behavior of the decision variables. Compared to the model by Omar and Yeo (2005), our model trades higher holding costs from carrying excess used items forward for better savings from repairing more used items.

The general optimality of the model is not investigated, so an immediate extension of this paper is to investigate in this direction. Another extension is to investigate the optimality of the sequence of production and repair runs (see Choi et al., 2007), which may be able to address the problem of carrying excess used items forward.

Acknowledgments

The authors thank the reviewers and the guest editor for their valuable and constructive comments.

References


