Analytical Proton Transfer Amplitude for Heavy Ion Induced Nuclear Reactions

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Abstract. Direct reactions between heavy ions have been studied widely using semi-classical theories. The Distorted Wave Born Approximation or DWBA has been extensively applied to analyse transfer reaction processes. Initial attempts to gain insights into the simple semi-classical parametrisation starting from the DWBA had focused mainly on neutron transfer reactions. An analytical formula for the semi-classical amplitude for the transfer of a single neutron between bound classical orbits states in heavy ion collisions that agrees well with the DWBA calculations has been successfully derived. In this paper, we have successfully derived the corresponding analytical expression for the proton transfer amplitude by using a technique analogous to the transfer of a single neutron between bound states. Our result reduces to the well known expression for the neutron transfer amplitude in the limit that the nuclear charge tends to zero.

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INTRODUCTION

In this paper, we present a method of deriving an analytical expression for the proton transfer amplitude between two bound states for the \( \ell = 0 \) case. We use a semi classical treatment that is similar in spirit to that adopted previously by Lo Monaco and Brink [1], and Bonarosco, Brink and Lo Monaco [2] in their seminal work on the transfer of a single neutron between bound states. In [1] an analytical expression for the neutron transfer amplitude was derived. In [2] a general expression formula for the nucleon transfer amplitude in momentum space was given. This formalism was also extended to include nuclear transfer reactions between heavy ions where the final state of the transferred nucleon is in the continuum [3]. More recently, differences that arise in reactions by neutron halo nucleus and a proton halo nucleus have also been studied [4]. However, our work differs in one important aspect as we consider the effect of the charge of the proton on the transfer amplitude formula for the transfer between bound states. Our starting point in deriving the transfer amplitude is the general nucleon transfer given in [2].

The Double Fourier Transform Of The Bound State Proton Wave Function

The general nucleon transfer amplitude formula in momentum space is [2]:

\[
A_{21} = \frac{i\hbar}{2\pi mv} \int_{-\infty}^{+\infty} dk_{y} \sqrt{k_{y}^{2} + \eta^{2}} 
\times \overline{\varphi}_{2}(d_{2}, k_{y}, k_{z}) \varphi_{1}^{*}(d_{1}, k_{y}, k_{z})
\]

(1)

where \( \overline{\varphi}_{\alpha}(d_{\alpha}, k_{y}, k_{z}) \) is the double Fourier transform of the bound state nucleon wave function in the initial (\( \alpha = 1 \)) and final (\( \alpha = 2 \)) states. Here \( m \) and \( v \) are respectively, the mass and velocity of the nucleon before transfer. The distance of closest approach of the projectile target) to the reference plane is given by \( d_{1} \) (\( d_{2} \)). Also

\[
\overline{\varphi}(x, k_{y}, k_{z}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy dz e^{-i[yk_{y} + zk_{z}]} \varphi(x, y, z)
\]

(2)

\[
\eta^{2} = k_{1}^{2} + \gamma_{1}^{2} = k_{2}^{2} + \gamma_{2}^{2},
\]

(3)
and \( \gamma_\alpha^2 = -\frac{2me}{h^2}, \alpha = 1, 2 \) \( \tag{4} \)

where \( \hbar k_1 \) and \( \hbar k_2 \) are the \( z \) components of the momentum of the transferred nucleon relative to the first and second nuclei, respectively \([2]\). Here \( \varepsilon \) represents the binding energy of the nucleon with \( \varepsilon < 0 \).

We begin by writing down the bound state wave function for the proton:

\[
\psi(r) = \frac{C e^{-\gamma r}}{r}(2\gamma)^\beta Y_{lm}(\theta, \phi) \tag{5}
\]

where

\[
C = C_\varepsilon (2\gamma)^\beta \tag{6}
\]

\[
\gamma^2 = -\frac{2\mu \varepsilon}{\hbar^2} \tag{7}
\]

\[
\beta = 2\varepsilon \sqrt{-\frac{\mu}{2\hbar^2}} \tag{8}
\]

and \( Y_{lm}(\theta, \phi) \) is a spherical harmonic. We interpret \( \beta \) as the “measure” of the strength of the nuclear charge when the proton is bound to the nucleus of the projectile. Next we obtain the expression for the double Fourier Transform of the bound state wave function for \( \ell = 0 \) by substituting Eq. (5) into Eq. (2) and replacing \( r^{\beta} e^{-\gamma r} \) by its Fourier Transform

\[
\frac{4\pi \Gamma(1 + \beta)}{(2\pi)^3} \int -\infty \int -\infty \int -\infty dq_x e^{i q_x x} \frac{d^3q e^{i (q_x x + q_y y + q_z z)}}{q^2 + q^2 + q^2} (1 + \beta)^{1/2} I(1 + \beta)^{1/2} \times \sin \left[ (1 + \beta) \tan^{-1} \left( \frac{q_x}{q_y} \right) \right] \tag{9}
\]

The resulting expression simplifies further to a single integral over \( q_x \) by the use of delta functions. If we define \([3]\)

\[
\gamma_x^2 = k_y^2 + k_z^2 + \gamma^2 \tag{10}
\]

we obtain

\[
\psi(x, k_y, k_z) = \frac{C_\varepsilon (2\gamma)^\beta \Gamma(1 + \beta)}{\sqrt{\pi}} I_{12} \tag{11a}
\]

with

\[
I = \int \frac{dq_x e^{i q_x x}}{-\infty \int -\infty \int -\infty} \frac{d^3q e^{i (q_x x + q_y y + q_z z)}}{q^2 + q^2 + q^2} (1/2) (1/2) I(1 + \beta)^{1/2} \times \sin \left[ (1 + \beta) \tan^{-1} \left( \frac{q_x}{q_y} \right) \right] \tag{11b}
\]

When \( \beta \) is set to zero Eq (11a) reduces as expected to

\[
C_\varepsilon e^{\gamma_x^2 x^2} \gamma_x \tag{12}
\]

which agrees with the neutron wave function for the \( \ell = 0 \) case \([3]\).

Eq (11b) can be solved analytically using the method of contour integrals. The result that we obtained for real values of \( \beta > 0 \) is given by the following formula:

\[
I = \frac{\gamma_x^2}{\gamma_x} \left( \gamma_x \right)^\beta \left( \gamma_x \right)^\beta \sum_{k=0}^{\infty} \frac{1}{k! (\beta - 2k)!} \times \left( -\frac{\gamma_x}{2\gamma_x^2}\right)^k K_\beta^{k+1/2}(\gamma_x^2 x^2) \tag{12}
\]

where \( K_\beta(x) \) denotes a modified Bessel function \([5]\). Note that the above formula is exact only for integer values of \( \beta \). However, it provides an approximation for non integer values. We have confirmed this by numerically integrating Eq (11a) and comparing the result obtained with Eq. 12 for arbitrary values of \( x, \beta, \gamma_x \) and \( \gamma \). The final expression for the bound state wave function is obtained by substituting \( I \) in Eq (11a) with Eq (12).

**Derivation Of The Analytical Expression For The Transfer amplitude**

For the \( \ell = 0 \) case, the expression for the transfer amplitude is:

\[
A_{12} = \frac{C_1}{\sqrt{4\pi}} \frac{C_2}{\sqrt{4\pi}} 2\pi \sqrt{|d_1 d_2|} \frac{i\hbar}{2\pi m v} (\gamma_{x,1})^\beta (\gamma_{x,2})^\gamma \times \sum_{K=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^K}{K} \frac{1}{k! (\beta - 2k)!} I_{12}, \tag{13}
\]

where
\[ I_i = \int_{-\infty}^{\infty} dk_y g(k_y) = \int_{-\infty}^{\infty} dk_y \left( k_y^2 + \eta^2 \right)^{p/2} \]
\[ \times K_{\alpha+1/2}(d_y \sqrt{k_y^2 + \eta^2})K_{\beta+1/2}(d_y \sqrt{k_y^2 + \eta^2}) \]
and 
\[ a = \beta_1 - K', b = \beta_2 - k, p = -(b + a) \]

As it stands, \( I_i \) is a rather difficult integral to handle analytically, mainly because of the presence of the two modified Bessel functions with complicated arguments. Nevertheless, we have discovered that integration can be done analytically if we treat the integrand as a simple Gaussian function of the form
\[ f(k_y) = A_1 e^{-A_2 k_y^2} \]

Here \( A_1 \) and \( A_2 \) are just the coefficients that can be determined using the following conditions \( f(0) = g(0) \), \( f^*(0) = g^*(0) \). The final analytical result after substituting \( A_1 \) and \( A_2 \) into \( f(k_y) \) and integrating over \( dk_y \) yields
\[ I_i = 2\pi \eta^{1-(b+a)}K_{\alpha+1/2}(\eta d_1)^{1/2}K_{\beta+1/2}(\eta d_2)^{1/2} \]
\[ \times \left\{ 2(b+a)K_{\alpha+1/2}(\eta d_1)K_{\beta+1/2}(\eta d_2) \right\} \]
\[ + \eta d_1 K_{\alpha+1/2}(\eta d_1)(K_{\alpha+1/2}(\eta d_2) + K_{\beta+1/2}(\eta d_2)) \]
\[ + \eta d_2 K_{\beta+1/2}(\eta d_2)(K_{\alpha+1/2}(\eta d_1) + K_{\beta+1/2}(\eta d_1)) \]
\[ \left( K_{\alpha+1/2}(\eta d_1) + K_{\beta+1/2}(\eta d_1) \right)^{-1/2} \]

We have verified numerically that Eq. 16 is an excellent approximation to Eq.14.

The Behavior Of The Proton Transfer Amplitude

For the \( \ell = 0 \) case, the proton transfer amplitude is found by substituting the result of Eq (16) in Eq (14). Let us check the behavior of our formula in the limit when the nuclear charges approach zero. As the charges \( \beta_1 \) and \( \beta_2 \) approach zero, the proton transfer amplitude simplifies to the following form:
\[ A_{12} = \frac{i \hbar}{mv} C_1 C_2 e^{-\eta(d_1 + d_2)} \frac{\pi}{\sqrt{2[1 + \eta(d_1 + d_2)]}} \]

The corresponding expression for the neutron transfer process is
\[ A'_{12} = \frac{i \hbar}{mv} C_1 C_2 K_0(\eta(d_1 + d_2)) \]

Likewise, \( \eta(d_1 + d_2) \) is large. By replacing the modified Bessel function with its asymptotic form
\[ e^{-\eta(d_1 + d_2)} \frac{\pi}{\sqrt{2[1 + \eta(d_1 + d_2)]}} \]
we obtain an expression identical to Eq (18).

CONCLUSION

In this paper, we have derived an analytical expression for the proton transfer amplitude between bound states for simple quasi elastic transfer reactions. We have shown that the formalism developed from the study of the single neutron transfer between bound states can be applied to the present proton case as well to obtain the transfer amplitude.

For the \( \ell = 0 \) case, we have demonstrated that our expression reduces to the familiar neutron result in the limit when the nuclear charges tend to zero. Work is in progress to generalize the proton transfer amplitude formula to \( \ell \neq 0 \).

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