Recursive formula to compute Zernike radial polynomials

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In optics, Zernike polynomials are widely used in testing, wavefront sensing, and aberration theory. This unique set of radial polynomials is orthogonal over the unit circle and finite on its boundary. This Letter presents a recursive formula to compute Zernike radial polynomials using a relationship between radial polynomials and Chebyshev polynomials of the second kind. Unlike the previous algorithms, the derived recurrence relation depends neither on the degree nor on the azimuthal order of the radial polynomials. This leads to a reduction in the computational complexity.

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Zernike polynomials are a set of basis functions that satisfy the orthogonality property on the continuous unit disk [1]. This set has found numerous applications in a variety of fields: optics [2], wavefront sensing [3], aberration theory [4], scaled pupils [5–7], and image processing [8,9]. However, direct computation of the Zernike polynomials, especially calculation of the factorial terms in the radial component, is time consuming. Many methods have been developed to speed up the computation of Zernike radial polynomials. Kintner [10] has proposed a recurrence relation for radial polynomials using four coefficients dependent on the degree and the azimuthal order of the radial components of the Zernike basis. Prata and Rusch [11] showed another recursive formula using only two dependent coefficients to compute the radial polynomials. However, some values for these methods must be directly computed from the definition of the Zernike radial polynomials, which increases excessive computation cost. It can be seen, in all computational methods, that the recurrence relations depend on degree and azimuthal order, with some different coefficients. Janssen and Dirksen [12] computed Zernike polynomials in the form of a discrete Fourier (cosine) transform that features advantages and efficiency over other methods in terms of computational aspects.

In this Letter, we propose a new recursive formula for computing Zernike radial polynomials. In general, Zernike polynomials are defined as

$$Z_n^m(r,\theta) = N_n^m R_n^m(r)\Theta(m\theta), \tag{1}$$

where N_n^m is the normalization factor given by

$$N_n^m = \sqrt{\frac{2(n+1)}{1+\delta_{m0}}}, \tag{2}$$

δ_{m0} is the Kronecker delta function (i.e., $\delta_{m0}=1$ for $m=0$ and $\delta_{m0}=0$ for $m\neq 0$). The radial polynomials $R_n^m(r)$ can be defined as follows:

$$R_n^m(r) = \sum_{k=0}^{\frac{n-|m|}{2}} (-1)^k \frac{(n-k)!}{k!\left(\frac{n+m}{2}-k\right)!\left(\frac{n-m}{2}-k\right)!} r^{n-2k}. \tag{3}$$

The triangular function $\Theta(m\theta)$ is given by

$$\Theta(m\theta) = \begin{cases} \cos(m\theta); & m\geq 0 \\ -\sin(m\theta); & m<0 \end{cases}. \tag{4}$$

The radial component of the Zernike polynomials defined above satisfies the following orthogonality relationship:

$$\int_0^1 R_n^m(r)R_k^m(r)r\mathrm{d}r = \frac{\delta_{nk}}{2(n+1)}. \tag{5}$$

The radial polynomials possess a double indexing scheme where n is the radial degree and m is the azimuthal order. The value of n is a nonnegative integer, and m is an integer that satisfies $n-|m|=$ even. There are several fast recursive methods for the calculation of radial polynomials. The algorithms include direct, Kintner and Prata methods [10,11,13].

The computation of radial polynomials using the definition of them in Eq. (3) is denoted hereafter as a direct method. This equation is heavily dependent on factorial terms. There are four factorial functions to be computed for each $R_m^n(r)$.

Kintner [10] has proposed the following recurrence relation based on the known properties of the Jacobi polynomials that uses polynomials of a varying low-degree n with a fixed azimuthal order of m to compute the radial polynomials:

$$R_n^m(r) = \frac{1}{K_1}[(K_2r^2+K_3)R_{n-2}^m(r) + K_4R_{n-4}^m(r)],$$
$$n = m+4, m+6, \ldots \tag{6}$$

where the coefficients K_1, K_2, K_3, and K_4 are given by [13]

$$K_1 = \frac{(n+m)(n-m)(n-2)}{2},$$
$$K_2 = 2n(n-1)(n-2),$$
$$K_3 = -m^2(n-1) - n(n-1)(n-2),$$
$$K_4 = \frac{-n(n+m-2)(n-m-2)}{2}. \tag{7}$$

This method should be initialized by specifying cases in which $n=m$ and $n-m=2$. Chong *et al.* [14] proposed for this

$$R_m^m(r) = r^m,$$
$$R_{m+2}^m(r) = (m+2)r^{m+2} - (m+1)r^m. \qquad (8)$$

Prata and Rusch [11] derived a three-terms recurrence relation to compute radial polynomials as

$$R_n^m(r) = rL_1 R_{n-1}^{|m-1|}(r) + L_2 R_{n-2}^m(r), \qquad (9)$$

where the coefficients L_1 and L_2 are

$$L_1 = \frac{2n}{m+n},$$
$$L_2 = \frac{m-n}{m+n}. \qquad (10)$$

Starting by the initial value, $R_0^0(r) = 1$, it is possible to generate all radial polynomials.

The authors in [14] presented a faster algorithm to compute radial polynomials based on another recurrence relation as follows:

$$R_n^{m-4}(r) = H_1 R_n^m(r) + \left(H_2 + \frac{H_3}{r^2}\right) R_n^{m-2}(r), \qquad (11)$$

where the coefficients H_1, H_2, and H_3 are given by

$$H_1 = \frac{m(m-1)}{2} - mH_2 + \frac{H_3(n+m+2)(n-m)}{8},$$
$$H_2 = \frac{H_3(n+m)(n-m+2)}{4(m-1)} + (m-2),$$
$$H_3 = \frac{-4(m-2)(m-3)}{(n+m-2)(n-m+4)}. \qquad (12)$$

It can be seen from the above equation that it is not applicable in cases where $n = m$ and $n - m = 2$. For these cases, we can use Eq. (8).

In all recursive algorithms to compute the Zernike radial polynomials, the coefficients depend on two basic factors, n and m, as the degree and the azimuthal order, respectively. In this Letter, we propose a new method avoids these dependencies. First, we apply Cormack's result that was used by Janssen and Dirksen [12] to find the integral representation of radial polynomials

$$R_n^m(r) = \frac{1}{2\pi}\int_0^{2\pi} U_n(r\cos\theta)\cos(m\theta)\mathrm{d}\theta, \qquad (13)$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind and satisfies the following recurrence relation (for $n \geq 2$):

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \qquad (14)$$

with $U_0(x) = 1$ and $U_1(x) = 2x$. The advantage of Eq. (13) is the separation ability of two factors, n and m, in radial polynomials. As we can see, the degree of radial polynomials is equal to the degree of Chebyshev polynomials of the second kind, and the azimuthal order equals the frequency of the cosine function. Set $x = r\cos\theta$ at Eq. (14) and multiply both sides by $\cos(m\theta)$ to get

$$U_n(r\cos\theta)\cos(m\theta) = (2r\cos\theta)U_{n-1}(r\cos\theta)\cos(m\theta) - U_{n-2}(r\cos\theta)\cos(m\theta). \qquad (15)$$

Using the product of cosine rules in the first term of the right-hand side of Eq. (15), we can simplify it as follows:

$$U_n(r\cos\theta)\cos(m\theta) = r[\cos(|m-1|\theta) + \cos((m+1)\theta)] \times U_{n-1}(r\cos\theta)\cos(m\theta) - U_{n-2}(r\cos\theta)\cos(m\theta). \qquad (16)$$

Now, both sides of Eq. (16) are integrated over θ with limits of integration 0 and 2π, and dividing by 2π, we have

$$\frac{1}{2\pi}\int_0^{2\pi} U_n(r\cos\theta)\cos(m\theta)\mathrm{d}\theta = r\left\{\frac{1}{2\pi}\int_0^{2\pi} U_{n-1}(r\cos\theta)\cos(|m-1|\theta)\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} U_{n-1}(r\cos\theta)\cos((m+1)\theta)\mathrm{d}\theta\right\} - \frac{1}{2\pi}\int_0^{2\pi} U_{n-2}(r\cos\theta)\cos(m\theta)\mathrm{d}\theta. \qquad (17)$$

Finally, using integral representation of the radial polynomials in Eq. (13) and applying it to each integral term in Eq. (17) leads to the following four-terms recursive formula to compute the Zernike radial polynomials:

$$R_n^m(r) = r[R_{n-1}^{|m-1|}(r) + R_{n-1}^{m+1}(r)] - R_{n-2}^m(r). \qquad (18)$$

With the initialization $R_0^0(r) = 1$, we can use Eq. (18) to compute all radial polynomials $R_n^m(r)$ with integer $n, m \geq 0$ such that $n - m$ is even and nonnegative, and $R_n^m(r) \equiv 0$ when $n < m$. Using an example of degree six, the computational flow of the proposed recursive method is illustrated in Fig. 1. In this flowchart, there are three groups of radial polynomials.

The first group is shown on the diagonal side of the chart and represents the case in which $n = m$. For this case, as we can see from the flow, the radial polynomial is connected to the previous polynomial of the same diagonal line using a blue arrow, and is multiplied by r. The second group of radial polynomials is located in

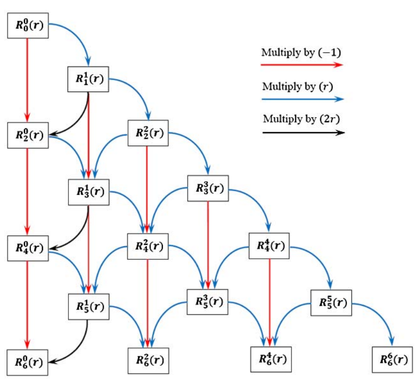

Fig. 1. Flow of proposed method to compute Zernike radial polynomials $R_n^m(r)$.

Table 1. Complexity Analysis of Radial Polynomials, $R_n^m(r)$, up to Degree n in Terms of A, Where $A = (\lceil(3/2) - (n/2)\rceil - 3)\lceil(3/2) - (n/2)\rceil + \lfloor n/2\rfloor^2 + \lfloor n/2\rfloor + 2n + 2$

Algorithm	Additions	Multiplications
Prata	$(3/2)A$	$(5/2)A$
Proposed	A	$(1/2)A$

the left-most column and satisfies $n = 0, 2, 4, \ldots$ and $m = 0$. For this case, the radial polynomials can be achieved by the two previous polynomials using multiplication factors of (-1) and $2r$, which are shown with red and black arrows, respectively. The rest of the radial polynomials are situated between the right-most diagonal line and the left-most column. In this case, $n \neq m$ and $m \neq 0$. The radial polynomials of this group are created by three previous polynomials using two multiplication factors of r and one multiplication factor of (-1), which is illustrated in Fig. 1. In all three aforementioned cases, only one unique recursive formula [Eq. (18)] is used, and this formula does not depend on the radial polynomials indices, n and m, whereas all prior discussed methods in this Letter are highly dependent on the parameters of radial polynomials in multiplication factors.

Table 1 shows the computational complexity of two recursive methods to compute radial polynomials. Symbols $\lceil.\rceil$ and $\lfloor.\rfloor$ are presented as the ceiling and floor functions. We do not consider the Kintner and q-recursive methods, because these two methods follow different recursion in computing radial polynomials, whereas the Prata and proposed recursive relations follow one unique formula for any given n and m. As we can see from Table 1, the number of additions in the proposed method up to degree n is $A = (\lceil(3/2) - (n/2)\rceil - 3)\lceil(3/2) - (n/2)\rceil + \lfloor n/2\rfloor^2 + \lfloor n/2\rfloor + 2n + 2$. Note that there are $\lfloor(n+2)^2/4\rfloor$ Zernike radial polynomials up to degree n. The number of multiplications to compute radial polynomials in the proposed method reduces to half the number of additions $(A/2)$, while the number of additions and multiplications increases to 1.5 and 5 times of A in Prata's method, respectively.

Figure 2 shows the number of required additions $(+)$ and multiplications $(\times)$ to compute the first 600 Zernike radial polynomials up to the 47th degree. From Fig. 2, the computation of radial polynomials using Prata's method up to the 40th degree requires 1320 additions and 2200 multiplications, whereas the proposed method would require 880 additions and 440 multiplications for the same degree of radial polynomials.

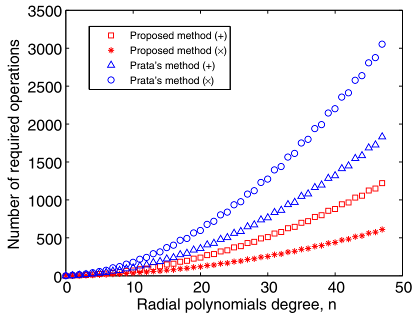

Fig. 2. Number of operations required to compute Zernike radial polynomials up to degree n.

In conclusion, the proposed algorithm shows a new recursive relationship with coefficients that are independent of radial degree n and azimuthal order m for calculation of Zernike radial polynomials. This reduces the computational complexity of radial polynomials, as the proposed method uses 80% of the number of multiplications and 33.3% of the number of additions as compared with Prata's algorithm.

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