Image reconstruction from a complete set of geometric and complex moments

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Abstract

An image can be reconstructed from the finite set of its orthogonal moments. Since geometric and complex moment kernels do not satisfy orthogonality criterion, direct image reconstruction using them is deemed to be difficult. In this paper, we propose a technique to reconstruct an image from either geometric moments (GMs) or complex moments (CMs). We utilize a relationship between GMs and Stirling numbers of the second kind. Then, by using the invertibility property of the Stirling transform, the original image can be reconstructed from its complete set of either geometric or complex moments. Further, based on previous works on blur effects on a moment domain and using the proposed reconstruction methods, a formulation is shown to obtain an estimated original image from the degraded image moments and the blur parameter. The reconstruction performance of the proposed methods on blur images is presented to validate the theoretical framework.

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1. Introduction

Moment functions computed from an image have been widely used as a basic feature descriptors in image analysis, pattern and object recognition, image classification, degraded signals classification, image watermarking and template matching [1–6]. Two-dimensional geometric moments (GMs), sometimes also called usual moments, were introduced by Hu [7]. He used the theory of algebraic invariants to derive a set of GM functions that are invariant with respect to translation, scaling as well as rotation. These types of moments not only provide measures of the shapes, such as volumes and surface areas, but also allow for the encoding of a shape with descriptors that are amenable to fast analysis such as database screening or pairwise shape comparison.

Complex moments (CMs) were introduced by Abu-Mostafa and Psaltis [8] as a simple and direct method to perform image normalization. However, kernel functions of GMs and CMs are not orthogonal. This makes the reconstruction of an image from its GMs or CMs quite difficult. Teague [9] has proposed three different approaches to solve the inverse problem of non-orthogonal moments. The first approach was the usage of characteristic function. He showed a relationship between the 2D Fourier transform of an image and its geometric moments by using the expansion of Taylor series. Then from the 2D Fourier transform, the original image can be obtained by the inversion formula. However, the length of order to be used for an accurate reconstruction may exceed the size of the image.

Using the Teagues' first approach, Ghorbel et al. [10] applied the discrete Fourier transform (DFT) as a characteristic function to reconstruct the original image using the approximated expansion of exponential function kernel up to a limited order of GMs. However, by taking the same number of GMs order as size of the image, this approximation causes error [11]. Ghorbel et al. [12] also used the
DFT approximation method for image reconstruction from a complete set of CMs. They proposed a systematic method to reconstruct an image from a finite set of its moment invariants by exploiting the link between the DFT of an image and its CMs.

Teague’s second method for determining an inverse GM transform is based on moment matching. Teague applied a continuous function as a polynomial to reconstruct the original image. This approach seems to be impractical as it requires the solution to an increasing number of coupled equations as higher order moments are considered.

The third approach to overcome the image reconstruction problem from a set of non-orthogonal moments is using orthogonal moments. Teague [9] showed that by replacing the monomial kernel of GMs with Legendre and Zernike continuous orthogonal polynomials, an image can be reconstructed from their orthogonal moments. Recently, a set of another continuous orthogonal moments is introduced such as Gaussian–Hermite moments [13], Bessel–Fourier moments [14] and Gegenbauer moments [15].

Mukundan et al. [16] and Yap et al. [17] introduced a set of discrete orthogonal moments based on the discrete Tchebichef polynomials and Krawtchouk polynomials, respectively. The development of these discrete orthogonal moments spurs the growth of other discrete orthogonal moments such as Hahn, dual-Hahn, Meixner, Charlier and Racah moments [18–21]. If we use the full set of Tchebichef or Krawtchouk moments, the original image can be recovered accurately.

Flusser et al. [11] propose a direct method to reconstruct an image using its computed geometric moments. As reported in [11], the algorithm produces exact reconstruction for images up to 11 × 11. For larger images, higher order moments as well as the kernel functions lose their precision and make it difficult for the image to be reconstructed exactly.

Though the geometric and complex moments are used as object descriptors, there has been substantial work done to link geometric and complex moments to point spread functions. In this regard, Flusser et al. [11] derived a general expression for degraded image moments in terms of the original image moments and the point-spread function (PSF). Additionally, they also showed a relationship that can be formulated between CMs of the degraded image caused by the Gaussian blur and the original image. In the case of GMs of the degraded image caused by the Gaussian blur, Liu and Zhang [22] derived an explicit formula for the GMs of the PSF.

In this paper, based on their contributions which lack the reconstruction of the image from GMs or CMs, we were motivated to solve the reconstruction directly from GMs and CMs. By using the relationship established in [11,22], we can formulate an inverse relationship between the original image and the blurred image. Firstly, we used the relationship of Stirling numbers [23] of the second kind and the kernels of GMs to generate the GMs. This paves the way to use the invertibility property of the Stirling transform to reconstruct an image from a complete set of GMs. Since CMs can be expressed in terms of GMs, we can also implement the same approach to reconstruct the original image from its CMs. It is worth noting, for an accurate reconstruction, the moments order has to be of the same size as the original image. We then show an application to restore original image from the blurred image using the derived reconstruction capabilities of either GMs or CMs and blur parameter.

This paper is organized as follows. In Section 2, a brief review of the related research and the results obtained are discussed. Section 3 gives the proposed methodology of GMs and CMs reconstruction. In this section, by using the Stirling transform, we show how we derive a new formulation for image reconstruction based on its GMs and CMs. The representation of the blurred image reconstruction using the proposed method is given in Section 4. In Section 5, the computational aspects of the proposed algorithm are outlined through experimental results which show the desirable features of the new method. Finally, Section 6 concludes the paper.

2. A brief review of the related works

Using GM as a basic tool for the purpose of image processing has several advantages over other orthogonal moments. The focus of most of the works has been on the orthogonal moments, but a lot of them have tried to describe the orthogonal moments in terms of GMs as a linear combination. In this section, some of the basic concepts are reviewed, and complementary ones are proposed.

An image is a real discrete 2D function with size N × M. The GM of order \((p, q)\) of an image, \(f(x, y)\) in the spatial domain is defined by

\[
m_{pq} = \sum_{x=1}^{N} \sum_{y=1}^{M} x^{p} y^{q} f(x, y). \tag{1}\]

The CMs of order \((p, q)\) of the same image in the spatial domain are defined by [8]

\[
C_{pq} = \sum_{x=1}^{N} \sum_{y=1}^{M} (x + iy)^{p} (x - iy)^{q} f(x, y) \tag{2}\]

where \(i = \sqrt{-1}\). The relationship between CMs and GMs can be obtained as follows [11]:

\[
C_{pq} = \sum_{k=0}^{p} \sum_{l=0}^{q} \binom{p}{k} \binom{q}{l} (-1)^{q-l} p^{l} q^{k-l} m_{k+l+p+q-k-l}. \tag{3}\]

The inverse relationship between GMs and CMs can be obtained as follows [11]:

\[
m_{pq} = \sum_{k=0}^{p} \sum_{l=0}^{q} \binom{p}{k} \binom{q}{l} (-1)^{q-l} \frac{p^{l} q^{k-l}}{2^{p+q} l!} C_{k+l+p+q-k-l}. \tag{4}\]

Eqs. (3) and (4) can be achieved from binomial expansion of the complex kernels in (2).

The discrete Fourier transform (DFT) of an image is defined by the following:

\[
F(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x, y) e^{-i2\pi(xu/N) + (vy/M)}. \tag{5}\]
The corresponding inverse discrete Fourier transform (IDFT) is defined by the following:

\[ f(x, y) = \frac{1}{NM} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u, v)e^{2\pi i (ux/N + vy/M)}. \]  

(6)

\[ F(u, v) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (-i2\pi)^{j+l} \frac{(u^j)}{j!} \frac{(v^l)}{l!} m_{jl}. \]  

(7)

From (7) we conclude that, \( m_{jl} \) is essentially the expansion coefficient of the \( iv^l \), whereas the inverse Fourier transform is given by the following:

\[ f(x, y) = \frac{1}{NM} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (-i2\pi)^{j+l} \frac{(u^j)}{j!} \frac{(v^l)}{l!} e^{2\pi i (ux/N + vy/M)} m_{jl}. \]  

(8)

Eq. (8) shows the image reconstruction from its infinite set of GMs via DFT. Ghorbel et al. [10] used the same algorithm to reconstruct the original image from GMs. The main drawback of this method is that the power series is only an approximation of the exponential function. If we take the same number of moment orders as image size, still there will be an error because of this approximation. They showed the image reconstruction of the binary 32 \( \times \) 32 letter ‘E’ using (8). Their results show the reconstruction error for 110 orders of GMs is 32 based on the middle threshold and for the same order the error is 6 based on the adaptive threshold.

If the number of GMs is not sufficient, the most affected frequencies are the highest ones. We need to remove the highest frequencies from the DFT spectrum to obtain an acceptable result. Therefore, we have two parameters to fix: the optimal GM order and the optimal threshold in the DFT domain.

2.2. Inverse CM using DFT

Ghorbel et al. [12] expressed the DFT of an image in terms of its CMs as follows:

\[ F(u, v) = \sum_{p=0}^{\infty} \frac{(-i2\pi)^p}{p!} \sum_{k=0}^{p} \binom{p}{k} \left( \frac{u+iv}{N} \right)^{p-k} \left( \frac{u-iv}{M} \right)^k C_{k,p-k}. \]  

(9)

\[ f(x, y) = \frac{1}{NM} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u, v)e^{2\pi i (ux/N + vy/M)}. \]  

(6)

2.1. Inverse GM using DFT

Teague [9] considered the Fourier transform as a characteristic function for an image and described it in terms of its ordinary moments. Using now DFT definition in (5), the relationship between the DFT and GMs is given by the following:

3. Proposed method for image reconstruction from non-orthogonal moments

As we have seen in the previous section, the DFT is a useful tool to recover the original image from its GMs or CMs, but because of the approximation of the exponential function, the error of this approximation is still too high. In this section, we use a different approach to reconstruct the original image from the same number of non-orthogonal moments as the number of image pixels.

3.1. Reconstruction from GMs

The basic idea is how we can relate the GM kernels which are in monomials form, \( x^p y^q \), with the Stirling number of the second kind. First, consider a 1D case. The generating functions for the Stirling numbers of the second kind \( S_2(p, k) \) are

\[ x^p = \sum_{k=0}^{p} k! S_2(p, k) \binom{x}{k} \]  

(10)

where \( x^p \) is the kernel of 1D GMs. By substituting (10) into the 1D form of (1), we can equate GMs in terms of the Stirling numbers of the second kind as follows:

\[ m_p = \sum_{x=1}^{N} \sum_{k=0}^{p} f(x)k! S_2(p, k) \binom{x}{k} \]  

(11)

\[ g(k) = \sum_{x=1}^{N} f(x)k! \binom{x}{k}. \]  

(12)

For two arbitrary sequences, \( g(\cdot) \) and \( h(\cdot) \) of length \( N \), using the Stirling transform which is an invertible relationship between the Stirling numbers of the first and second kinds \([23, 24]\), we have the following:

\[ h(p) = \sum_{k=0}^{p} S_2(p, k) g(k) \leftrightarrow g(i) = \sum_{k=0}^{i} S_1(i, k) h(k). \]  

(13)

Applying (13) into (11):

\[ g(i) = \sum_{k=0}^{i} S_1(i, k) m_k. \]  

(14)

Substituting (12) into (14) and changing \( k \) to \( p \), (14) can be rewritten as follows:

\[ \sum_{x=1}^{N} f(x) \binom{x}{i} = \frac{1}{i!} \sum_{p=0}^{i} S_1(i, p) m_p. \]  

(15)

To recover the original signal \( f(x) \) from (15), we first establish the following transform between two sequences \( f(x) \) and \( \phi(i) \) of length \( N \) using the combinations of \( \binom{\cdot}{i} \) and \( \binom{\cdot}{\cdot} \):

\[ \phi(i) = \sum_{x=1}^{N} f(x) \binom{x}{i}. \]  

(16)

By expanding (16), we get \( N \)-simultaneous equations. Solving these equations in reverse order, the sequence \( f(x) \)
can be obtained from \( \phi(i) \) as follows:

\[
f(x) = \sum_{i=1}^{N} ( -1 )^{i-x} \binom{i}{x} \phi(i).
\]

(17)

Using (16) and (17), the original sequence, \( f(x) \), can be derived from (15) as follows:

\[
f(x) = \sum_{p=0}^{N-1} \sum_{i=0}^{N-1} \left( -1 \right)^{i-x} \binom{i}{x} S_i (i+1, p+1) m_p.
\]

(18)

Full reconstruction of the original signal can be obtained using (18) with a complete set of GMs. In the 2D case, for an \( N \times M \) pixel spatial pattern \( f(x,y) \), reconstruction from a complete set of GMs can be generalized as follows:

\[
f(x,y) = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \left( -1 \right)^{q+y} \binom{q+y}{q} S_q (j+1, q+1) m_{pq}.
\]

(19)

3.2. Reconstruction from CMs

From (9), the DFT of an image can be expressed in terms of its CMs. However, errors are introduced because the Taylor series is only an approximation of the exponential function. In the previous subsection, we proposed an accurate algorithm using the Stirling numbers of the first kind to reconstruct the original image from its complete set of GMs. Using the relationship between GMs and CMs as described in (4), and using (19), image reconstruction from a complete set of CMs is given by the following:

\[
f(x,y) = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \left( -1 \right)^{q+y} \binom{q+y}{q} S_q (j+1, q+1) m_{pq}
\]

\[
\times \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \left( \frac{p}{k} \right) \left( \frac{q}{l} \right) \left( -1 \right)^{k+l} \binom{k+l}{k} \binom{k+l}{l} C_{k+l,p+q-k-l}. \]

(20)

4. Blurred image reconstruction using the proposed methods

In this section, we establish a relationship between the proposed method for image reconstruction and the GMs and CMs of the blurred image. Blurring can be described by a convolution

\[
g(x,y) = f(x,y) \ast h(x,y)
\]

(21)

where \( f(x,y) \) is an original image, \( g(x,y) \) is a degraded image and \( h(x,y) \) is a point-spread function (PSF) of the image system. For a Gaussian PSF \( h(x,y) = (1/2\pi \sigma^2)e^{-\left(x^2+y^2\right)/2\sigma^2} \). Flusser and Suk [25], obtained classical relations for the GMs and CMs of the blurred images in terms of the original image and PSF moments as follows:

\[
m_p^{(g)} = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \left( \frac{p}{k} \right) \left( \frac{q}{l} \right) m_k^{(b)} m_l^{(f)}
\]

(22)

and

\[
c_p^{(g)} = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \left( \frac{p}{k} \right) \left( \frac{q}{l} \right) c_k^{(b)} c_l^{(f)}
\]

(23)

where \( m_p^{(g)}, c_p^{(g)}, m_p^{(b)}, c_p^{(b)}, m_p^{(f)} \) and \( c_p^{(f)} \) are the GMs and CMs of the degraded image, PSF and the original image, respectively. Liu and Zhang [22] derived an explicit formula for the GMs of the Gaussian PSF which is simplified here as follows:

\[
m_p^{(q)} = \begin{cases} 2^{-(p+q)/2} \frac{\Gamma(l)}{\Gamma(l/2)} \sigma^{p+q} & \text{if } p \text{ and } q \text{ are even} \\ 0 & \text{otherwise}. \end{cases}
\]

(24)

Flusser et al. [11] obtained an expression for the CMs of the Gaussian PSF as follows:

\[
c_p^{(q)} = (2\pi)^{1/2} \sigma^p \delta_{pq}
\]

where \( \delta_{pq} \) is the Kronecker delta function (i.e. \( \delta_{pq} = 1 \) for \( p = q \) and \( \delta_{pq} = 0 \) for \( p \neq q \)). Substituting (24) into (22) and taking the inverse transform to relate the original image GMs in terms of the blurred image GMs yields

\[
m_p^{(q)} = \sum_{j=0}^{N-1} \sum_{l=0}^{M-1} \left( \frac{p}{k} \right) \left( \frac{q}{l} \right) \left( -2 \right)^{-k-l} \frac{k!l!}{(k+l)!} \sigma^{k+l} m_{p-kq-l}^{(g)}
\]

(25)

Applying the same procedures to derive CMs and using Eqs. (23) and (25), the CMs of the original image are given by the following:

\[
c_p^{(q)} = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \left( \frac{p}{k} \right) \left( \frac{q}{l} \right) \left( -2 \pi^2 \right)^{1/2} \delta_{pq} c_p^{(g)}
\]

(26)

Now, we apply the proposed method to reconstruct the original image from a full set of GMs or CMs of a blurred image which is degraded by a Gaussian blur. In this paper, the focus is to show a relationship of the estimated original image with the degraded image moments and Gaussian blur parameter, \( \sigma \). Substituting (26) into (19) results in

\[
\hat{f}_s(x,y)_{\text{GM}} = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} A_{ij}^{(q)} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} B_{k,l}^{(q)} (i,j) m_{k+l,p+q-k-l}
\]

(28)

where

\[
A_{ij}^{(q)} = \left( \frac{p}{k} \right) \left( \frac{q}{l} \right) \left( -1 \right)^{i+j-x-y} \binom{i}{x} \binom{j}{y} S_i (i+1, p+1) S_j (j+1, q+1)
\]

(29)

and

\[
B_{k,l}^{(q)} = \left( \frac{p}{k} \right) \left( \frac{q}{l} \right) \left( -2 \right)^{-(k+l)/2} \frac{k!l!}{(k+l)!} \sigma^{k+l}
\]

(30)

In (28), the indexes of \( k \) and \( l \) take the even integer values. Similarly, we can apply the same approach to reconstruct the image from a full set of CMs in terms of \( \sigma \) as follows:

\[
\hat{f}_s(x,y)_{\text{CM}} = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} A_{ij}^{(q)}
\]

\[
\sum_{k=0}^{N-1} \sum_{l=0}^{M-1} D_{k,l}^{(q)} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} E_{u,v,k,l}^{(q)} (i,j) m_{k+l,p+q-k-l}
\]

(31)

where

\[
D_{k,l}^{(q)} = \left( \frac{p}{k} \right) \left( \frac{q}{l} \right) \left( -1 \right)^{i+j-x-y} \binom{i}{x} \binom{j}{y} \left( 2^p + q \frac{(2)}{(\sqrt{3})^q} \right)
\]

(32)
and

\[ E_{u,v,k,l}^{p,q}(\sigma) = \binom{k+l}{u} \binom{p+q-k-l}{v} (-2\sigma^2)^u!v! \delta_{ul}. \]  

(33)

5. Experimental studies

In this section we will illustrate the performance of the proposed approaches for real images. Throughout all the experiments, it can be seen that the proposed methods obtain fairly good results. We present the test data and results used to validate the theoretical framework presented above, and also establish the feature representation ability of GMs and CMs through image reconstruction. The image reconstruction capability using GMs and CMs is shown by some experiments. We illustrated an example to show the stability of the proposed method of a 1D signal reconstruction. A comparison in terms of computational time between the proposed method and the DFT approach is discussed for different image sizes. Here a full comparison of our method with that reported work by [10] is not done for the following reasons: (1) The main advantage of the technique presented by Ghorbel et al. [10] is its ability to reconstruct the original image from GMs. As noted by the authors, this technique is an approximation and needs to use higher order of GMs or CMs to reach an accurate reconstructed image. Whereas, the orthogonal moments such as Tchebichef moments would give an exact result when the moment orders touch the size of original image. So, their method is defined only if the higher orders of GMs or CMs are used. On the contrary, Ghorbel’s algorithm is less efficient than the matrix implementation of GMs or CMs using by Flusser’s matrix approach [11]. (2) By formulating the Gaussian blur in the geometric and complex moments domain, it is possible to recover the blurred images from a full set of GMs or CMs using the proposed methods. Fig. 1 shows the flowchart of the blurred image recovery from their GMs or CMs. We used the statistical normalization image reconstruction error (SNIRE) defined in [26] to measure the performance of the blurred image reconstruction:

\[ E_n = \frac{\sum_{x=0}^{N-1} \sum_{y=0}^{M-1} |\hat{f}(x,y) - \tilde{f}(x,y)|^2}{\sum_{x=0}^{N-1} \sum_{y=0}^{M-1} |f(x,y)|^2}. \]  

(34)

BRISQUE is a no-reference image quality assessment model based on natural scene statistic and operates in the spatial domain. It uses features from scene statistics of locally mean subtracted contrast normalized (MSCN) luminance coefficients and the pairwise products of MSCN. The features are then fed into support vector machine regressor (SVR) to quantify possible losses of ‘naturalness’ in the image due to the presence of different types of distortions [27].

5.1. Image reconstruction from GMs and CMs

In Section 3, we have proved that it is possible to recover the original image from its complete set of GMs or CMs by means of Eq. (19) or (20). In this experiment, two different images were used. The first image is a binary 32 × 32 letter ‘E’ and shown in Fig. 2(a). The second image is a gray-scale 512 × 512 ‘Lena’ and displayed in Fig. 2(b). By using (19), both binary and gray-scale images were recovered from their GMs up to 32 and 512 orders, respectively. Fig. 2(c) and (d) shows the reconstructed images up to each size of images. Eq. (20) is a combination of (19) and (4) to reconstruct the original image from its CMs up to size of image. Since, the relationship between the GMs and CMs is exact, the image reconstruction error of complete set of CMs should be zero. The recovered images from CMs are illustrated in Fig. 2(e) and (f).

5.2. Stability of the proposed methods

Since the image reconstruction capability from GMs using (19) possesses the separation property, we consider only 1D case of this equation that is mentioned in (18). There is no approximation in the proof process of (18), when parameters \( p \) and \( i \) are reached to the image size \( (N-1) \), we can recover any signal from its limited GMs \( (m_0, m_1, \ldots, m_{N-1}) \). In other words, by using all \( N \) moments

![Fig. 1. Reconstruction algorithm using full set of GMs or CMs for blurred images.](image-url)
of any signal, it is possible to reconstruct the original signal using the Stirling numbers of the first kind exactly. Here, we illustrate the reconstruction stability with a simple example. Assume that a length-4 sequence \( f(x) \) has the random values of 8, 3, 12 and 5 for \( x = 0, 1, 2 \) and 3, respectively

\[
\begin{align*}
8; & \quad x = 0 \\
3; & \quad x = 1 \\
12; & \quad x = 2 \\
5; & \quad x = 3 \\
0 & \quad \text{otherwise}
\end{align*}
\]

It is clear, the first four GMs of the assumed sequence are given by the following:

\[
m_0 = 28, \quad m_1 = 70, \quad m_2 = 208, \quad \text{and} \quad m_3 = 676
\]

Using the proposed method, we need to use only four aforementioned moments to recover the original sequence, \( f(x) \), exactly. By using the matrix representation of (18) and substituting \( N=4 \), we can write

\[
\begin{bmatrix}
f(0) \\
f(1) \\
f(2) \\
f(3)
\end{bmatrix} =
\begin{bmatrix}
4 & -13 & \frac{1}{2} & \frac{1}{3} \\
-6 & \frac{10}{2} & -4 & \frac{1}{2} \\
4 & -7 & \frac{7}{2} & \frac{1}{3} \\
-1 & \frac{11}{4} & -1 & \frac{1}{6}
\end{bmatrix}
\begin{bmatrix}
m_0 \\
m_1 \\
m_2 \\
m_3
\end{bmatrix} =
\begin{bmatrix}
8 \\
3 \\
12 \\
5
\end{bmatrix}
\]

If we use the DFT approach [10] to get the same exact result, because of the Taylor series approximation of the exponential term, at least should take 60 GMs of the assumed sequence to derive the DFT coefficients of the original signal,

\[
f(x) = \frac{1}{4} \sum_{x = 0}^{3} \sum_{i = 0}^{59} \frac{1}{i!} (-1)^{2xk} e^{2\pi k/4} m_i
\]

Comparing with the DFT approach, the proposed method has the stability criterion when used the complete set of GMs, but the DFT method requires a large number of moment orders to create the accurate DFT coefficients.

5.3. Speed performances of the reconstructed images

Since the only competitor against our method is the DFT approach, we can compare the elapsed time to reconstruct an image from its GMs or CMs. As we mentioned earlier, to get the same result of image reconstruction from the DFT algorithm and the proposed method, the DFT method requires so many additions to compute the image pixel values. Recalling the 1D example of Section 5.2, the upper limits of double summations of proposed method are limited up to size of signal,

\[
f(x) = \sum_{p = 0}^{3} \sum_{i = 0}^{3} \frac{(-1)^{i-x}}{i!} S_1(i+1, p+1)m_p.
\]

Whereas, the upper limit of the second summation of the DFT algorithm is not limited to the size of signal and for this case, it is 59 which is satisfied from the exact reconstruction of the original signal (Eq. (38)). On the other hand, for the DFT algorithm, the kernel of summations is used complex numbers and it is necessary to compute the power complex function, \((-j2\pi k/N)^i\) and the complex exponential term, \(e^{2\pi k/4}m_i\). For the proposed method, the kernel of double summations is real and we compute the binomial function, \(\binom{i}{x}\) and the Stirling numbers of the first kind, \(S_1(i+1, p+1)\), which is easier than the kernel of the DFT algorithm. Because of separability property of GMs,
the reconstruction of a 2D signal followed the same algorithms. Table 1 shows the reconstruction error and CPU elapsed time for proposed methods and the DFT algorithms using binary letter ‘E’ and gray-scale ‘Lena’ images that were used in the first experiment. This figure illustrates both reconstruction errors and computational times from GMs and CMs.

5.4. Blurred image reconstruction from GMs

In the first part of these experiments, the proposed method is tested on reconstruction of degraded images from a complete set of GMs. The test images are blurred by the given Gaussian blur kernels. Table 2 illustrates four test images from LIVE database [28]: light house, painted

| Table 1 | Reconstruction error and CPU elapsed time for the proposed and DFT algorithms using binary and gray-scale images of Fig. 2(a) and (b). |
|---|---|---|---|---|---|---|
| Recons. | Original image and max order of reconstruction | Proposed method | | DFT approach |
| | | Recons. error | CPU elapsed time (s) | Recons. error | CPU elapsed time (s) |
| From GMs | | | | | |
| ![E](image) | Max order = 32 | 0.0 | 4.2386 | 0.0 | 23.4051 | 0.0 | 31.9810 | 0.0 | 31.9810 |
| ![E](image) | Max order = 32 | 0.0 | 5.0012 | 0.0 | 31.9810 | 0.0 | 31.9810 |
| From CMs | | | | | |
| ![E](image) | Max order = 32 | 0.0 | 4.2386 | 0.0 | 23.4051 | 0.0 | 31.9810 | 0.0 | 31.9810 |
| ![E](image) | Max order = 32 | 0.0 | 5.0012 | 0.0 | 31.9810 | 0.0 | 31.9810 |

| Table 2 | Image reconstruction using GMs for test images of various sizes (below the original images) with different estimated \(\sigma\) (below the reconstructed images) and their corresponding SNIRE and BRISQUE. |
|---|---|---|
| Blur image | Reconstructed images | Original image |
| ![image](image) | ![image](image) | ![image](image) |
| \(\sigma = 14.99\) | 0.2 | 2.76 | 5.34 | 7.76 | 10.34 | 12.93 | 480 \(\times\) 720 |
| SNIRE | 0.2368 | 0.2269 | 0.2169 | 0.2056 | 0.1820 | 0.0299 | – |
| BRISQUE \(\times 10^2\) | 0.9617 | 0.9441 | 0.9121 | 0.8560 | 0.7180 | 0.0930 | – |
| ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) |
| \(\sigma = 7.67\) | 0.2 | 1.38 | 2.56 | 3.54 | 4.92 | 6.29 | 768 \(\times\) 512 |
| SNIRE | 0.2181 | 0.2089 | 0.1940 | 0.1823 | 0.1589 | 0.0363 | – |
| BRISQUE \(\times 10^2\) | 0.9378 | 0.9091 | 0.8302 | 0.7690 | 0.6615 | 0.1588 | – |
| ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) |
| \(\sigma = 11.33\) | 0.2 | 1.26 | 2.02 | 4.29 | 6.81 | 9.08 | 768 \(\times\) 512 |
| SNIRE | 0.2721 | 0.2578 | 0.2216 | 0.1426 | 0.1136 | 0.0581 | – |
| BRISQUE \(\times 10^2\) | 0.9299 | 0.9022 | 0.8296 | 0.6491 | 0.5570 | 0.0168 | – |
| ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) | ![image](image) |
| \(\sigma = 5.83\) | 0.2 | 0.70 | 1.40 | 2.80 | 4.43 | 5.36 | 480 \(\times\) 720 |
| SNIRE | 0.1267 | 0.1181 | 0.0964 | 0.0767 | 0.0581 | 0.0168 | – |
| BRISQUE \(\times 10^2\) | 0.9206 | 0.8863 | 0.7691 | 0.5617 | 0.3162 | 0.0564 | – |
house, monarch and caps, the sizes of which were 480/C2 720, 768/C2 512, 768/C2 512 and 480/C2 720, respectively. Each image is degraded by a Gaussian PSF with standard deviation (σ) of 14.99, 7.67, 11.33 and 5.83, respectively. This table shows the deblurred images using degraded image reconstruction from their GMs. We begin with an initial standard deviation value. By using Eq. (28), we can reconstruct the image, \( f_s(x, y) \). The reconstructed image is then compared with the original image, \( f(x, y) \). If the error, \( \varepsilon_n \), is greater than an adaptive threshold value, \( T \), the value of \( \sigma \) is increased for the next iteration. This process continues until the error is less than the value of \( T \).

### 5.5. Blurred image reconstruction from CMs

In the second part of this experiment, we used another four test images from LIVE database [28]: sailing, parrots, rapids and ocean, the sizes of which were 480 × 720, 768 × 512, 768 × 512 and 768 × 512, respectively. Each image is degraded by a Gaussian PSF with standard deviation (σ) of 11.33, 7.67, 7.67 and 5.83, respectively (Table 3). The same method of the image reconstruction from CMs is used here to find the blur parameter of the blurred image.

Table 3

<table>
<thead>
<tr>
<th>Blur image</th>
<th>Reconstructed images</th>
<th>Original image</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ = 11.33</td>
<td>0.2 1.44 3.30 5.56 7.62 9.48</td>
<td>480 × 720</td>
</tr>
<tr>
<td>SNIRE</td>
<td>0.1624 0.1550 0.1442 0.1252 0.0977 0.0171</td>
<td></td>
</tr>
<tr>
<td>BRISQUE ( \times 10^2 )</td>
<td>0.9462 0.9215 0.8743 0.7829 0.6610 0.0951</td>
<td></td>
</tr>
</tbody>
</table>

 besides using SNIRE to measure the error between the original and reconstructed images, BRISQUE is also used as another means of error measurement. From Eqs. (28) and (31), it can be seen that computation of the reconstructed image from CMs is more computationally intensive than the reconstructed image from GMs.

### 6. Conclusion

In this paper, a new approach has been proposed for image reconstruction from geometric and complex moments. This approach has been developed using the Stirling numbers of the first and second kinds and uses the property of orthogonality conditions. Unlike the traditional methods which used the DFT or other orthogonal domains, this algorithm is very simple and possesses less complexity. In the discussion on inverse problem of GMs and CMs, it is illustrated how this algorithm could recover some part of the original image up to the desired order accurately.

In order to evaluate the performance of the new reconstruction approach from GMs and CMs, the binary and gray scale images have been applied to test the accuracy of the proposed methods. It is further shown that the proposed reconstruction methods contain the stability criterion. Additionally, to illustrate the proposed
method abilities for reconstructing of blurry images, eight gray scale images from LIVE database have been applied in two experiments. The factors, SNIRE and BRISQUE, showed the error measurement and the image quality in these analyses, respectively. The proposed algorithm can be used like three solutions which Teague \cite{8} has suggested for a perfect image reconstruction from usual and non-orthogonal moments such as GMs and CMs.

There are a few papers on image recovery from GMs or CMs, each of which yields specific advantages. Finding other type of applications such as invariant family or 3D object recognition of shape position and orientation is a major direction for further research on the proposed algorithm.

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