On Classification of Real Hypersurfaces in a Complex Space Form with $\eta$-recurrent Shape Operator

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Abstract. In this paper, we classify real hypersurfaces in a non-flat complex space with $\eta$-recurrent shape operator.

1. Introduction

Let $M_n(c)$ be an $n$-dimensional complete and simply connected non-flat complex space form with complex structure $J$ of constant holomorphic sectional curvature $4c$, i.e., it is either a complex projective space $\mathbb{C}P^n$ (for $c > 0$), or a complex hyperbolic space $\mathbb{C}H^n$ (for $c < 0$).

Suppose $M$ is a connected real hypersurface in $M_n(c)$ and $N$ is a unit normal vector field of $M$. Let $\xi = -JN$ be the structure vector field and $A$ the shape operator on $M$. A Hopf hypersurface $M$ in $M_n(c)$ is characterized by the condition that the structure vector field $\xi$ is principal, i.e., $A\xi = \alpha \xi$, and it can be shown that this principal curvature $\alpha$ is a constant.

Typical examples of Hopf hypersurfaces are those with constant principal curvatures, nowadays, so-called real hypersurfaces of type $A_1$, $A_2$, $B$, $C$, $D$ and $E$ (resp. of type $A_0$, $A_1$, $A_2$ and $B$) in $\mathbb{C}P^n$ (resp. in $\mathbb{C}H^n$) (cf. [14], [12]). These real hypersurfaces can be expressed as tubes of constant radius over certain holomorphic or totally real submanifolds, and a self-tube in the ambient space (cf. [1], [2], [5]).

Other than these Hopf hypersurfaces, another example of real hypersurfaces in $M_n(c)$ are the class of ruled real hypersurfaces. Ruled real hypersurfaces in $M_n(c)$ are characterized by having a one-codimensional foliation whose leaves are complex totally geodesic hyperplanes in $M_n(c)$. The geometry of ruled real hypersurfaces in $M_n(c)$ was studied in [10].

The study of real hypersurfaces in a non-flat complex space form has been an active field in the past few decades. One of the first results is the non-existence of real hypersurfaces with parallel shape operator $A$, i.e., $\nabla A = 0$, where $\nabla$ is the Levi-Civita connection of $M$. 

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This fact is an immediate consequence of the Codazzi equation of such a real hypersurface. Motivated by this, Kimura and Maeda [6] studied the weaker notion of $\eta$-parallelism. The shape operator $A$ is said to be $\eta$-parallel if it satisfies the following condition:

$$\langle (\nabla_X A) Y, Z \rangle = 0$$

for any $X, Y, Z \in \Gamma(D)$, where $D := \text{Span}\{\xi\}^\perp$, called the (maximal) holomorphic distribution on $M$. A number of results concerning real hypersurfaces with $\eta$-parallel shape operator have been obtained (cf. [6], [7], [8], [13]). In particular, a complete classification of real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator was proved in [8] (cf. Theorem 4).

In another way to weaken the parallelism, Hamada [3] studied the recurrence of the shape operator of real hypersurfaces in $\mathbb{C}P^n$. The shape operator $A$ is said to be recurrent if $\nabla A = A \otimes \omega$ for some 1-form $\omega$ in $M$. It was showed in [3] that the recurrence is also too strong to be satisfied by the shape operator of real hypersurfaces in $\mathbb{C}P^n$.

The shape operator $A$ is said to be $\eta$-recurrent if there is a 1-form $\omega$ on $M$ such that

$$\langle (\nabla_X A) Y, Z \rangle = \omega(X)\langle AX, Y \rangle$$

for any $X, Y, Z \in \Gamma(D)$. The $\eta$-parallelism and recurrence can be considered as special cases of $\eta$-recurrence. Hopf hypersurfaces in $M_n(c)$ with $\eta$-recurrent shape operator were classified in [4, 11].

**THEOREM 1** ([4, 11]). Let $M$ be a Hopf hypersurface in $M_n(c)$, $n \geq 3$, $c \neq 0$. Then the shape operator $A$ is $\eta$-recurrent if and only if $M$ is locally congruent to one of the following spaces:

(a) For $c > 0$:

(A1) a tube over hyperplane $\mathbb{C}P_{n-1}$;
(A2) a tube over totally geodesic $\mathbb{C}P_k$, where $1 \leq k \leq n - 2$;
(B) a tube over complex quadric $Q_{n-1}$.

(b) For $c < 0$:

(A0) a horosphere;
(A1) a geodesic hypersphere or a tube over hyperplane $\mathbb{C}H_{n-1}$;
(A2) a tube over totally geodesic $\mathbb{C}H_k$, where $1 \leq k \leq n - 2$;
(B) a tube over totally real hyperbolic space $\mathbb{RH}^n$.

The purpose of this paper is to improve the above theorem and classify real hypersurfaces in $M_n(c)$ with $\eta$-recurrent shape operator, i.e., we prove the following theorem.

**THEOREM 2.** Let $M$ be a real hypersurface in $M_n(c)$, $n \geq 3$, $c \neq 0$. Then its shape operator $A$ is $\eta$-recurrent if and only if $M$ is locally congruent to a ruled real hypersurface or one of the following spaces:

(a) For $c > 0$:

(A1) a tube over hyperplane $\mathbb{C}P_{n-1}$;
(A2) a tube over totally geodesic $\mathbb{C}P_k$, where $1 \leq k \leq n - 2$;
(B) a tube over complex quadric $Q_{n-1}$.

(b) For $c < 0$:

(A)
(A0) a horosphere;
(A1) a geodesic hypersphere or a tube over hyperplane $CH_{n-1}$;
(A2) a tube over totally geodesic $CH_k$, where $1 \leq k \leq n - 2$;
(B) a tube over totally real hyperbolic space $RH^n$.

2. Preliminaries

In this section we shall recall some fundamental identities and known results in the theory of real hypersurfaces in a complex space form and fix some notations.

Let $M$ be a connected real hypersurface isometrically immersed in $M_n(c)$, $n \geq 3$, $N$ a unit normal vector field on $M$ and $\langle \cdot, \cdot \rangle$ the Riemannian metric on $M$. We define a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = \langle \xi, X \rangle,$$

for any $X \in \Gamma(TM)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1. \quad (1)$$

Denote by $\nabla$ the Levi-Civita connection and $A$ the shape operator on $M$. Then

$$\nabla_X \phi Y = \eta(Y)AX - \langle AX, Y \rangle \xi, \quad \nabla_X \xi = \phi AX \quad (2)$$

for any $X, Y \in \Gamma(TM)$.

Let $R$ be the curvature tensor of $M$. Then the equations of Gauss and Codazzi are given respectively by

$$R(X, Y)Z = c\{\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle \phi Y, Z \rangle \phi X - \langle \phi X, Z \rangle \phi Y - 2\langle \phi X, Y \rangle \phi Z + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY \}
$$

$$A_{\nabla X}Y - (\nabla Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y \rangle \xi\}.$$

This following lemma is needed in the next section.

**Lemma 3 ([9]).** Let $M$ be a non-Hopf real hypersurface in $M_n(c)$, $n \geq 3$, $c \neq 0$. Suppose $A\xi = \alpha\xi + \beta U$ and $AU = \beta\xi + \gamma U$, where $\beta = ||\phi A\xi|| (> 0)$ and $U = -\beta^{-1}\phi^2 A\xi$. If there exists a unit vector field $Z \perp \xi, U, \phi U$ such that $AZ = \lambda Z$ and $A\phi Z = \lambda\phi Z$, then

$$(\lambda - \gamma)(\lambda^2 - \alpha\lambda - c) - \beta^2\lambda = 0.$$

Finally, we state without proof the following result concerning real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator.
THEOREM 4 ([8]). Let $M$ be a real hypersurface in $M_n(c)$, $n \geq 3$, $c \neq 0$. Then its shape operator $A$ is $\eta$-parallel if and only if $M$ is locally congruent to a ruled real hypersurface or one of the following spaces:

(a) For $c > 0$:
- $(A_1)$ a tube over hyperplane $CP_{n-1};$
- $(A_2)$ a tube over totally geodesic $CP_k$, where $1 \leq k \leq n - 2$;
- $(B)$ a tube over complex quadric $Q_{n-1}$.

(b) For $c < 0$:
- $(A_0)$ a horosphere;
- $(A_1)$ a geodesic hypersphere or a tube over hyperplane $CH_{n-1};$
- $(A_2)$ a tube over totally geodesic $CH_k$, where $1 \leq k \leq n - 2$;
- $(B)$ a tube over totally real hyperbolic space $RH^n$.

3. Proof of Theorem 2

Let $M$ be a real hypersurface in $M_n(c)$, $n \geq 3$, with $\eta$-recurrent shape operator, i.e.,

$$((\nabla_X A)Y, Z) = \omega(X)(AY, Z)$$

(3)

for any $X, Y, Z \in \Gamma(D)$, where $\omega$ is a 1-form on $M$. By virtue of Theorem 1, we only need to consider the non-Hopf case. In this case, $\beta := \|\phi A\xi\| > 0$ and we may define a unit vector field $U := -\beta^{-1}\phi^2 A\xi$. It suffices to prove that $A$ is $\eta$-parallel or $\omega = 0$ according to Theorem 4. Suppose to the contrary that $\omega \neq 0$. Let $W'$ be the vector field dual to $\omega$ and $b := \|\phi W'\|$. Then $b > 0$ at some open subset $G$ of $M$. Since we only study local geometric property, we may identify $M$ with this open subset $G$ and define a unit vector field $W = -b^{-1}\phi^2 W'$. Hence (3) can be rewritten as

$$((\nabla_X A)Y, Z) = b(X, W)(AY, Z)$$

(4)

for any $X, Y, Z \in \Gamma(D)$. It follow from the Codazzi equation, (4) and the fact $b > 0$ that

$$\langle X, W \rangle \langle AY, Z \rangle = \langle Y, W \rangle \langle AX, Z \rangle.$$

By putting $X = Z = W$ in the above equation, we obtain $\phi AW = \gamma \phi W$, where $\gamma = \langle AW, W \rangle$. Hence, after putting $X = W$ in the above equation, we have

$$\langle AY, Z \rangle = \gamma \langle Y, W \rangle \langle Z, W \rangle$$

(5)

for any $Y, Z \in \Gamma(D)$.

By (4) and (5), we see that $\gamma = 0$ is equivalent to $\omega = 0$. Hence, we get $\gamma \neq 0$. By differentiating covariantly both sides of the above equation in the direction of $X \in \Gamma(D)$; with the help of (1), (2) and 5, we have
\[(\nabla_X A)Y, Z - \beta(Y, \phi A X)\langle U, Z \rangle - \beta(Z, \phi A X)\langle Y, U \rangle\]
\[= d\gamma(X)\langle Y, W \rangle\langle Z, W \rangle + \gamma\langle Y, \nabla_X W \rangle\langle Z, W \rangle + \gamma\langle Y, W \rangle\langle Z, \nabla_X W \rangle. \quad (6)\]

By using (4) and (5), the above equation becomes
\[\gamma b\langle X, W \rangle\langle Y, W \rangle\langle Z, W \rangle - \gamma\beta\langle Y, \phi W \rangle\langle X, W \rangle\langle U, Z \rangle - \gamma\beta\langle Z, \phi W \rangle\langle X, W \rangle\langle Y, U \rangle\]
\[= d\gamma(X)\langle Y, W \rangle\langle Z, W \rangle + \gamma\langle Y, \nabla_X W \rangle\langle Z, W \rangle + \gamma\langle Y, W \rangle\langle Z, \nabla_X W \rangle. \quad (7)\]

If we let \(Y = Z = W\) in the above equation, then \(\gamma b\langle X, W \rangle = d\gamma(X)\), for any \(X \in \Gamma(D)\).

With this fact, (7) reduces to
\[-\beta\langle Y, \phi W \rangle\langle X, W \rangle\langle U, Z \rangle - \beta\langle Z, \phi W \rangle\langle X, W \rangle\langle Y, U \rangle\]
\[= \langle Y, \nabla_X W \rangle\langle Z, W \rangle + \langle Y, W \rangle\langle Z, \nabla_X W \rangle. \quad (8)\]

Next, by letting \(X = W, Y = Z = \phi W\) in (8), we have \(\langle \phi W, U \rangle = 0\). Finally, after putting \(X = W\) and \(Z = \phi W\) in (8), yields \(-\beta U = \langle \phi W, \nabla_W W \rangle W\). Since both \(U\) and \(W\) are unit vector fields, we may, without loss of generality, assume that \(U = W\). This, together with (5), yields \(A U = \beta \xi + \gamma U\) and \(A Z = 0\), for any \(Z \perp U, \xi\). According to Lemma 3, we can see that \(\gamma = 0\). This contradicts the fact that \(\gamma \neq 0\) and so the proof is completed.

The following result has been obtained in [7].

**THEOREM 5 ([7]).** Let \(M\) be a real hypersurface in \(M_n(c)\), \(n \geq 3, c \neq 0\). Then \(M\) satisfies
\[(\nabla_X A)Y = \{-c\langle \phi X, Y \rangle + \eta(A Y)\langle X, \phi A \xi \rangle + \eta(A X)\langle Y, \phi A \xi \rangle + \varepsilon \langle (\phi A - A \phi)X, Y \rangle\}\xi \]
for any \(X, Y \in \Gamma(D)\), where \(\varepsilon\) is a constant, if and only if \(M\) is locally congruent to one of the spaces stated in Theorem 2.

By Theorem 2 and Theorem 5, we can characterize the \(\eta\)-recurrence of \(A\) by an expression of the covariant derivative of \(A\) on the holomorphic distribution.

**COROLLARY 6.** Let \(M\) be a real hypersurface in \(M_n(c)\), \(n \geq 3, c \neq 0\). Then the following are equivalent:

1. the shape operator \(A\) is \(\eta\)-recurrent;
2. \((\nabla_X A)Y = \{-c\langle \phi X, Y \rangle + \eta(A Y)\langle X, \phi A \xi \rangle + \eta(A X)\langle Y, \phi A \xi \rangle + \varepsilon \langle (\phi A - A \phi)X, Y \rangle\}\xi, \)
for any \(X, Y \in \Gamma(D)\), where \(\varepsilon\) is a constant;
3. \(M\) is locally congruent to one of the spaces stated in Theorem 2.

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