Real hypersurfaces in a complex space form with $\eta$-parallel shape operator

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Abstract In this paper, we classify the real hypersurfaces in a non-flat complex space form with $\eta$-parallel shape operator.

Keywords Complex space forms · Hopf hypersurfaces · Ruled real hypersurfaces · $\eta$-parallel shape operator

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1 Introduction

One of the most fertile subjects in differential geometry is the theory of real hypersurfaces in a complex space form. A complex space form can be considered as a natural generalization of a space of constant curvature with a complex structure $J$. Its complex structure $J$ imposes significant restrictions on the geometry of its real hypersurfaces. For instance, there exist no totally umbilical real hypersurfaces and real hypersurfaces with parallel shape operator in a non-flat complex space form. Weaker ideas of “$\eta$-” conditions, such as $\eta$-umbilical and $\eta$-parallelism, have been considered due to these restrictions.

One of the main lines of research deal with characterizations of real hypersurfaces in a complex space form under these restrictions. This paper is a contribution in this direction. Our objective is to classify real hypersurfaces in a non-flat complex space form with $\eta$-parallel shape operator.
We shall now review some results in the theory of real hypersurfaces in a non-flat complex space form in order to describe this more precisely and state our main theorem.

By $M_n(c)$ we denote an $n$-dimensional complete and simply connected non-flat complex space form with constant holomorphic sectional curvature $4c$, i.e., it is either a complex projective space $\mathbb{C}P^n$ or a complex hyperbolic space $\mathbb{C}H^n$ (according to as the holomorphic sectional curvature $4c$ is positive or negative). Let $M$ be a connected real hypersurface in $M_n(c)$ and $N$ a unit normal vector field of $M$. We denote by $\Gamma(V)$ the module of all differentiable sections on the vector bundle $V$ over $M$. Then the complex structure $J$ of $M_n(c)$ induces an almost contact metric structure $(\phi, \xi, \eta, \langle, \rangle)$ on $M$, i.e.,

$$JX = \phi X + \eta(X)\xi, \quad J\xi = -\eta, \quad \eta(X) = \langle \xi, X \rangle$$

for any $X \in \Gamma(TM)$.

A real hypersurface is said to be Hopf if the structure vector field $\xi$ is principal, i.e., $A\xi = \alpha \xi$, for some function $\alpha$ on $M$. In 1973, Takagi [16] classified homogeneous real hypersurfaces in $\mathbb{C}P^n$ into six classes of Hopf hypersurfaces with constant principal curvatures. Having extended the results of Cecil and Ryan in [3], Kimura [6] showed that the converse of Takagi’s result is also true, i.e.,

**Theorem 1** ([6]) Let $M$ be a Hopf hypersurface of $\mathbb{C}P^n$, $n \geq 2$, with constant principal curvatures. Then $M$ is locally congruent to one of the following hypersurfaces:

$(A_1)$ geodesic spheres,
$(A_2)$ tubes over totally geodesic $\mathbb{C}P^p$, for $p \in \{1, \ldots, n-2\}$,
$(B)$ tubes over complex quadrics $Q^{n-1}$ and totally real projective space $\mathbb{R}P^n$,
$(C)$ tubes over the Serge embedding of $\mathbb{C}P^1 \times \mathbb{C}P^m$, where $2m + 1 = n$ and $n \geq 5$,
$(D)$ tubes over the Plücker embedding of the complex Grassmann manifold $G_{2,5}$, where $n = 9$.
$(E)$ tubes over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$, where $n = 15$.

**Remark 1** Let $M_1$ and $M_2$ be two submanifolds of a Riemannian manifold $\tilde{M}$. We say that $M_1$ is locally congruent to $M_2$ if there is an isometry $f$ on $\tilde{M}$ such that $f(M_1)$ is an open subset of $M_2$. An analogous result in $\mathbb{C}H^n$ has also been proven by Berndt [2].

**Theorem 2** ([2]) Let $M$ be a Hopf hypersurface of $\mathbb{C}H^n$, $n \geq 2$, with constant principal curvatures. Then $M$ is locally congruent to one of the following hypersurfaces:

$(A_0)$ horospheres,
$(A_1)$ geodesic spheres and tubes over totally geodesic complex hyperbolic hyperplanes,
$(A_2)$ tubes over totally geodesic $\mathbb{C}H^p$, for $p \in \{1, \ldots, n-2\}$,
$(B)$ tubes over totally real hyperbolic space $\mathbb{R}H^n$.

In what follows, by real hypersurfaces of type $A$, we mean of type $A_1$, $A_2$ (resp. of type $A_0$, $A_1$, $A_2$) for $c > 0$ (resp. for $c < 0$).

From a result of Tashiro and Tachibana [18], we see that there are no totally umbilical hypersurfaces in $M_n(c)$. A weaker notion of $\eta$-umbilical hypersurfaces was hence considered. A real hypersurface $M$ is said to be totally $\eta$-umbilical if the shape operator satisfies

$$AX = aX + b\eta(X)\xi$$
for any $X \in \Gamma(TM)$, where $a$ and $b$ are functions on $M$. The classification of totally $\eta$-umbilical real hypersurfaces in a complex projective space has been obtained by Kon (cf. [8]), and then extended by Montiel (cf. [12]) to complex hyperbolic ambient space.

The shape operator $A$ is said to be $\eta$-parallel if

$$\langle (\nabla_X A) Y, Z \rangle = 0$$

for any $X, Y, Z \in \Gamma(D)$, where $D := \text{span}(\xi) \perp$ is the holomorphic distribution on $M$ and $\nabla$ is the Levi-Civita connection on $M$. The $\eta$-parallelism condition on the shape operator was first introduced by Kimura and Maeda in [7]. The study of this condition was motivated by the non-existence of real hypersurfaces in $M_n(c)$ with parallel shape operator.

In the same paper, Kimura and Maeda classified Hopf hypersurfaces in a complex projective space with $\eta$-parallel shape operator. Soon after, this result was extended to the setting of real hypersurfaces in a complex hyperbolic space (cf. [15]).

**Theorem 3** ([7,13,15]) Let $M$ be a Hopf hypersurface in $M_n(c), n \geq 2$. Then $M$ has $\eta$-parallel shape operator if and only if $M$ is locally congruent to a real hypersurface of type $A$ or $B$.

Another example of real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator is the class of ruled real hypersurfaces. Ruled real hypersurfaces in $M_n(c)$ are characterized by having a one-codimensional foliation whose leaves are totally geodesic complex hypersurfaces in $M_n(c)$ (cf. [10]).

With an additional assumption on the integrability of the holomorphic distribution $D$, real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator appeared to be the ruled real hypersurfaces.

**Theorem 4** ([1,7]) Let $M$ be a real hypersurface in $M_n(c), n \geq 3$. Suppose $M$ satisfies the following two conditions:

1. $\phi(\phi A + A\phi)\phi = 0$, i.e., the holomorphic distribution $D$ is integrable;
2. the shape operator $A$ is $\eta$-parallel.

Then $M$ is locally congruent to a ruled real hypersurface.

A number of results concerning real hypersurfaces in a non-flat complex space form with $\eta$-parallel shape operator has been obtained (for instance, [1,4,5,9,14], etc). In particular, we have the following theorem.

**Theorem 5** ([9]) Let $M$ be a real hypersurface in $M_n(c), n \geq 3$, with $\eta$-parallel shape operator $A$. If $\phi A\phi$ and $\phi^2 A\phi^2$ commute then $M$ is locally congruent to a ruled real hypersurface, or a real hypersurface of type $A$ or $B$.

The complete classification of real hypersurfaces in a non-flat complex space form with $\eta$-parallel shape operator remains open up to this point. It is interesting to note that the real hypersurfaces appearing in the list of these characterizations are those of type $A$, $B$ and ruled real hypersurfaces. This raises the following question: Does the class of real hypersurfaces in a non-flat complex space form with $\eta$-parallel shape operator consist of those of type $A$, $B$ and ruled real hypersurfaces? In this paper, we shall answer this question affirmatively. More precisely, we shall improve Theorem 5 into the following.

**Theorem 6** Let $M$ be a real hypersurface in a complex space form $M_n(c), n \geq 3$. Then the shape operator $A$ is $\eta$-parallel if and only if $M$ is locally congruent to a ruled real hypersurface, or a real hypersurface of type $A$ or $B.
This paper is organized as follows. In Sect. 2 we fix some notations and review some known results on real hypersurfaces in $M_n(c)$. We characterize real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator under the assumption that $\phi A\phi$ and $\phi^2 A\phi^2$ share an eigenvector in $D$ in the next section. Section 4 is devoted to a characterization of ruled real hypersurfaces in $M_n(c)$. In Sect. 5 we prove the nonexistence of real hypersurfaces with $\eta$-parallel shape operator in $M_n(c)$ with no common eigenvectors in $D$ for $\phi A\phi$ and $\phi^2 A\phi^2$. Finally, the classification of real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator is proved in Sect. 6.

2 Preliminaries

In this section we shall fix some notations and recall some known results on real hypersurfaces in a non-flat complex space form.

Let $M$ be a connected real hypersurface in $M_n(c)$. Then the induced almost contact metric structure $(\phi, \xi, \eta, \langle, \rangle)$ on $M$ has the following properties

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1 \quad (1)$$

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y \rangle\xi, \quad \nabla_X \xi = \phi AX \quad (2)$$

for any $X, Y \in \Gamma(TM)$, where $\nabla$ is the Levi-Civita connection on $M$. Let $R$ be the curvature tensor of $M$. Then the equations of Gauss and Codazzi are given respectively by

$$R(X, Y)Z = c\{\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle \phi Y, Z \rangle \phi X - \langle \phi X, Z \rangle \phi Y$$

$$-2\langle \phi X, Y \rangle \phi Z + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y \rangle\xi\}.$$ 

The second order covariant derivative $\nabla^2 A$ on the shape operator $A$ is defined by

$$(\nabla_X Y)A = \nabla_X ((\nabla_Y A)Z) - (\nabla_{\nabla_X Y} A)Z - (\nabla_Y A)\nabla_X Z.$$ 

The following lemma characterizes ruled real hypersurfaces in $M_n(c)$.

**Lemma 1** ([11]) Let $M$ be a real hypersurface in $M_n(c), n \geq 2$. Then $M$ is a ruled real hypersurface if and only if $\phi A\phi = 0$, or equivalently $\langle AX, Y \rangle = 0$, for any $X, Y \in \Gamma(D)$.

The following lemma plays a key role in this paper.

**Lemma 2** ([9]) Let $M$ be a real hypersurface in $M_n(c)$ with $\eta$-parallel shape operator $A$, and let $F := \nabla_\xi A$. Then

$$c\{\langle Y, AZ \rangle \langle X, W \rangle - \langle X, AZ \rangle \langle Y, W \rangle$$

$$+ \langle \phi Y, AZ \rangle \langle \phi X, W \rangle - \langle \phi X, AZ \rangle \langle \phi Y, W \rangle - 2 \langle \phi X, Y \rangle \langle \phi AZ, W \rangle$$

$$- \langle Y, Z \rangle \langle X, AW \rangle + \langle X, Z \rangle \langle Y, AW \rangle$$

$$- \langle \phi Y, Z \rangle \langle \phi X, AW \rangle + \langle \phi X, Z \rangle \langle \phi Y, AW \rangle + 2 \langle \phi Y, Y \rangle \langle \phi Z, AW \rangle\}$$

$$+ \langle AY, AZ \rangle \langle AX, W \rangle - \langle AX, AZ \rangle \langle AY, W \rangle$$

$$- \langle AY, Z \rangle \langle AX, AW \rangle + \langle AX, Z \rangle \langle AY, AW \rangle$$

$$= c\{\langle Z, \phi AY \rangle \langle \phi X, W \rangle + \langle W, \phi AY \rangle \langle \phi X, Z \rangle$$
Finally, by the Ricci identity
\[ (Z, \phi AX)\langle \phi Y, W \rangle - (W, \phi AX)\langle \phi Y, Z \rangle \]
\[ + (Y, \phi AX)\langle FZ, W \rangle + (Z, \phi AX)\langle FY, W \rangle + (W, \phi AX)\langle FZ, Y \rangle \]
\[ - (X, \phi AY)\langle FZ, W \rangle - (Z, \phi AY)\langle FX, W \rangle - (W, \phi AY)\langle FZ, X \rangle \]
for any \( X, Y, Z, W \in \Gamma(D) \).

**Proof** For any \( Y, Z, W \in \Gamma(D) \), by differentiating the following equation covariantly
\[ ((\nabla_Y A)Z, W) = 0 \]
in the direction of \( X \in \Gamma(D) \), we obtain
\[ ((\nabla^2_{XY} A)Z + (\nabla_{\nabla_X Y} A)Z + (\nabla_Y A)\nabla_X Z, W) + ((\nabla_Y A)Z, \nabla_X W) = 0. \]
From the \( \eta \)-parallelism condition and (2), the above equation reduces to
\[ ((\nabla^2_{XY} A)Z, W) = (Y, \phi AX)\langle (\nabla_\xi A)Z, W \rangle + (Z, \phi AX)\langle (\nabla_Y A)\xi, W \rangle \]
\[ + (W, \phi AX)\langle (\nabla_Y A)Z, \xi \rangle. \]
Furthermore, by using the Codazzi equation, the above equation becomes
\[ ((\nabla^2_{XY} A)Z, W) = (Y, \phi AX)\langle FZ, W \rangle + (Z, \phi AX)\langle FY, W \rangle - c\langle \phi Y, W \rangle \]
\[ + (W, \phi AX)\langle FY, Z \rangle - c\langle \phi Y, Z \rangle. \]
Finally, by the Ricci identity \( (R(X, Y)A)Z = (\nabla^2_{XY} A)Z - (\nabla^2_{YX} A)Z \) and the above equation, we obtain the lemma.

### 3 Real hypersurfaces with common eigenvectors in \( D \) for both \( \phi^2 A\phi^2 \) and \( \phi A\phi \)

In the rest of this paper, we suppose that \( M \) is a connected real hypersurface in a complex space form \( M_n(c) \), \( c \neq 0 \), \( n \geq 3 \), with \( \eta \)-parallel shape operator \( A \).

We shall begin the proof of Theorem 6 in this section. As a first step, we shall give a slight improvement of Theorem 5.

Note that the commutativity of \( \phi A\phi \) and \( \phi^2 A\phi^2 \), as stated in Theorem 5, is equivalent to the simultaneous diagonalizability of \( \phi A\phi \) and \( \phi^2 A\phi^2 \) by orthonormal eigenvectors \( \xi, E_1, E_2, \ldots, E_{2n-2} \) on \( T_x M \) at each point \( x \in M \). We shall show that the commutativity of \( \phi A\phi \) and \( \phi^2 A\phi^2 \) is replaceable by a weaker condition “\( \phi A\phi \) and \( \phi^2 A\phi^2 \) share an eigenvector \( E \in D_x \) at each point \( x \in M \)”, i.e., we have

**Theorem 7** Let \( M \) be a real hypersurface in \( M_n(c) \), \( n \geq 3 \), with \( \eta \)-parallel shape operator \( A \). If for each \( x \in M \), there is a unit vector \( E \in D_x \) such that \( E \) is an eigenvector of both \( \phi A\phi \) and \( \phi^2 A\phi^2 \) then \( M \) is locally congruent to a ruled real hypersurface, or a real hypersurface of type \( A \) or \( B \).

**Proof** We write \( E = E_1 \). Since \( \phi^2 A\phi^2 \) is self-adjoint and, both \( \xi \) and \( E_1 \) are orthonormal eigenvectors, we can extend to an orthonormal base formed by eigenvectors \( E_0 = \xi, E_1, E_2, \ldots, E_{2n-2} \) of \( \phi^2 A\phi^2 \) corresponding to the eigenvalues \( \lambda_0 = 0, \lambda_1, \ldots, \lambda_{2n-2} \) (with not necessarily distinct \( \lambda_k \)'s). Then we have \( \phi^2 A\phi^2 E_k = \lambda_k E_k \). From this, together with (1), we obtain
\[ AE_k = \beta_k \xi + \lambda_k E_k; \quad k \in \{1, \ldots, 2n-2\} \]
where \( \beta_k = \eta(AE_k) \).
Now suppose to the contrary that \( \phi A \phi \) and \( \phi^2 A \phi^2 \) cannot be diagonalized simultaneously. Then there is some \( k \) for which \( E_k \) is not an eigenvector of \( \phi A \phi \) and hence \( \phi E_k \) is not an eigenvector of \( \phi^2 A \phi^2 \). Consequently, there is a positive integer \( p < n - 1 \) such that \( \dot{E}_{p+j} = \phi E_j \), \( j \in \{1, \ldots, p\} \) and \( \phi E_t \) is not an eigenvector of \( \phi^2 A \phi^2 \), for \( t \in \{2p+1, \ldots, 2n-2\} \). Hence, we may write

\[
AE_j = \beta_j \xi + \lambda_j E_j, \quad \phi E_j = \tilde{\beta}_j \xi + \tilde{\lambda}_j E_j; \quad j \in \{1, \ldots, p\}
\]

\[
AE_t = \beta_t \xi + \lambda_t E_t; \quad t \in \{2p+1, \ldots, 2n-2\}.
\]

\[
A \phi E_t - \eta(A \phi E_t) \xi - (A \phi E_t, \phi E_t) \phi E_t \neq 0; \quad t \in \{2p+1, \ldots, 2n-2\}. \tag{3}
\]

In the following, we consider \( j \in \{1, \ldots, p\} \). By substituting \( Z = \phi E_j \), \( W = E_j \), and \( X, Y \perp E_j, \phi E_j \) in Lemma 2, we obtain

\[
2c(\phi X, Y)(\tilde{\lambda}_j - \lambda_j) = \langle F \phi E_j, E_j \rangle ((A + A \phi) X, Y). \tag{4}
\]

First of all, suppose that \( \langle F \phi E_j, E_j \rangle \neq 0 \). Since \( \text{span}\{E_j, \phi E_j\}^\perp \) is invariant by \( \phi A \phi \), it follows from (4) that

\[
A \phi X = \eta(A \phi X) \xi + \frac{2c(\tilde{\lambda}_j - \lambda_j)}{\langle F \phi E_j, E_j \rangle} \phi X - \phi AX; \quad X \perp E_j, \phi E_j.
\]

But this contradicts (3). Hence

\[
\langle F \phi E_j, E_j \rangle = 0. \tag{5}
\]

From (4), we see that \( \tilde{\lambda}_j - \lambda_j = 0 \). On the other hand, by putting \( X = \phi E_j \), \( Y = E_j \), and \( Z, W \perp E_j, \phi E_j \) in Lemma 2, we obtain

\[
2c(\phi Z - \phi AZ, W) = (\tilde{\lambda}_j + \lambda_j) \langle F Z, W \rangle. \tag{6}
\]

If \( \tilde{\lambda}_j = \lambda_j = 0 \) then by making use of the fact that \( \text{span}\{E_j, \phi E_j\}^\perp \) is invariant by \( \phi A \phi \) and (6), we have \( A \phi X = \eta(A \phi X) \xi + \phi AX \), for \( X \perp E_j, \phi E_j \). From this, together with (3), we obtain a contradiction. Hence, we assume that

\[
\tilde{\lambda}_j = \lambda_j \neq 0. \tag{7}
\]

In what follows, we consider \( t \in \{2p+1, \ldots, 2n-2\} \). By putting \( Z = W = E_t \) in (6), we obtain \( \langle FE_t, E_t \rangle = 0 \). On the other hand, if we substitute \( Y = Z = W = E_t \) in Lemma 2, then

\[
\lambda_t \phi (A^2 E_t - (A^2 E_t, E_t) E_t) = 0.
\]

Now, by substituting \( Z = W = E_t \) in Lemma 2 and by using the above equation, we can see that

\[
\langle A \phi E_t, Y \rangle (c \phi E_t + F E_t, X) = \langle A \phi E_t, X \rangle (c \phi E_t + F E_t, Y); \quad X, Y \in D_t.
\]

From (3), we see that \( A \phi E_t \neq 0 \) and so the above equation implies that

\[
\phi (c \phi E_t + F E_t - f_i A \phi E_t) = 0, \tag{8}
\]

for some constant \( f_i \). Finally, by using (7), (8) and letting \( Z = E_t \) in (6), we have \( \langle (c - \lambda_j f_i) A \phi E_t - c(\lambda_t - \lambda_j) \phi E_t, W \rangle = 0 \), for \( W \perp E_j, \phi E_j \). Since \( A \phi E_t, \phi E_t \in \text{span}\{E_j, \phi E_j\}^\perp \), we conclude that

\[
\phi \left( (c - \lambda_j f_i) A \phi E_t - c(\lambda_t - \lambda_j) \phi E_t \right) = 0.
\]
It follows from (3) that we get \( c - \lambda_j f_t = \lambda_t - \lambda_j = 0 \). From the above observation, we obtain
\[
AX = \eta(AX)\xi + \lambda_1 X, \quad X \in D_x.
\]
But this contradicts (3) and by Theorem 5, the proof is completed.

4 A characterization for ruled real hypersurfaces

Hopf hypersurfaces with \( \eta \)-parallel shape operator in \( M_n(c) \) have been completely classified in Theorem 3. We only have to focus on those real hypersurfaces on which \( \xi \) is not principal.

Suppose that the open set \( M_0 := \{ x \in M : \beta := ||\phi A\xi|| \neq 0 \} \) is nonempty. Then for each \( x \in M_0, \xi \) is not principal and so there exists a unit vector \( U \in D_x \) such that
\[
A\xi = \alpha\xi + \beta U, \quad (\alpha := \eta(A\xi)).
\]

In this section, we shall prove a characterization for ruled real hypersurfaces in \( M_n(c) \) under the \( \eta \)-parallelism of the shape operator and the assumption that \( U \) is an eigenvector for \( \phi^2 A\phi^2 \).

**Lemma 3** Let \( M \) be a real hypersurface in a complex space form \( M_n(c), n \geq 3 \), with \( \eta \)-parallel shape operator, and let \( x \in M_0 \). Suppose \( M \) satisfies the following hypotheses:

1. \( AU = \beta\xi + \lambda U, \) for some \( \lambda \in \mathbb{R}, \) and
2. \( A\phi U = \tilde{\lambda}\phi U + \tilde{\delta}\phi H, \) for some unit vector \( H \in D_x \) with \( H \perp U \) and \( \tilde{\lambda}, \tilde{\delta} \in \mathbb{R} \) with \( \tilde{\delta} \neq 0. \)

Then

(a) \( FU \in \text{span}\{\phi U, \phi H, \xi\}, \)
(b) \( \tilde{\lambda}(FU, \phi H) = c\tilde{\delta} + \tilde{\delta}(FU, \phi U), \)
(c) \( (F\phi H, \phi H) = 0, \)
(d) \( AH = \tau H, \) for some \( \tau \in \mathbb{R}, \)
(e) \( c\tilde{\delta} = \tau(FU, \phi H) \) \( (\neq 0), \)
(f) \( (F\phi U, \phi U) = 0, \)
(g) \( c\tilde{\delta} = \lambda(F\phi U, H) \) \( (\neq 0), \)
(h) \( (F\phi U, \phi H) = 0, \)
(i) \( A\phi H = \tilde{\delta}\phi U + \tilde{\tau}\phi H, \) for some \( \tilde{\tau} \in \mathbb{R}, \)
(j) \( c + \lambda\tilde{\tau} - \tilde{\delta}^2 = 0, \)
(k) \( FH \in \text{span}\{\phi U, \phi H, \xi\}, \)
(l) \( \tilde{\tau}(F\phi U, H) = c\tilde{\delta} + \tilde{\delta}(FH, \phi H). \)

**Proof** First, by letting \( X = \phi H, Y = Z = W = U \) in Lemma 2 and with the understanding of \( \tilde{\delta} \neq 0, \) we get \( \langle FU, U \rangle = 0. \) Hence after putting \( Z = W = U \) in Lemma 2 and taking account of \( \langle FU, U \rangle = 0, \) yields
\[
\langle A\phi U, Y \rangle \langle FU + c\phi U, X \rangle = \langle A\phi U, X \rangle \langle FU + c\phi U, Y \rangle
\]
for any \( X, Y \in D_x. \) Since \( \tilde{\delta} \neq 0, \) Assertion (a) can be deduced from this equation when we consider \( Y = \phi H. \) Moreover, by putting \( X = \phi H \) and \( Y = \phi U \) in the above equation, we obtain Assertion (b).
Next, by putting $Y = U$, $Z = W = H$ in Lemma 2 and taking account of Assertion (a), we get

$$4c \langle AH, \phi H \rangle \phi U = \langle FH, H \rangle \{\lambda + \tilde{\lambda}\phi U + \tilde{\delta} \phi H\}.$$ 

Since $\phi H \perp \phi U$ and $\tilde{\delta} \neq 0$, we have

$$\langle A\phi H, H \rangle = \langle FH, H \rangle = 0. \quad (9)$$

If we substitute $Y = U$ and $Z = W = \phi H$ in Lemma 2 then

$$-2c\tilde{\delta} H = \langle F\phi H, \phi H \rangle \{\lambda + \tilde{\lambda}\phi U + \tilde{\delta} \phi H\} - 2\langle F\phi H, U \rangle AH.$$ 

By using (9) and the above equation, we obtain assertions (c), (d) and (e).

By putting $Y = H$ and $Z = W = \phi U$ in Lemma 2, we get

$$-2c\tilde{\delta} \langle U, X \rangle = \langle F\phi U, \phi U \rangle \{\tau \phi H + A\phi H, X\} - 2\lambda\langle F\phi U, H \rangle \langle U, X \rangle$$

for any $X \in \mathcal{D}_x$. If we first put $X = \phi U$, follow by $X = U$ in this equation then we obtain assertions (f) and (g).

Now, let $Y = \phi H$ and $Z = W = \phi U$ in Lemma 2 and taking account of Assertion (i), we have

$$c\tilde{\delta} \phi U + \langle A^2 \phi U, \phi H \rangle A\phi U - \tilde{\delta} A^2 \phi U = \lambda \langle F\phi U, \phi H \rangle U.$$ 

Since all $\phi U$, $A\phi U$ and $A^2 \phi U \perp U$, we have Assertion (h) and

$$c\tilde{\delta} \phi U + \langle A^2 \phi U, \phi H \rangle A\phi U - \tilde{\delta} A^2 \phi U = 0. \quad (10)$$

By taking inner product of (10) with $X \perp \phi U, \phi H$, we have

$$0 = \langle A^2 \phi U, X \rangle = \tilde{\delta} \langle A\phi U, X \rangle = \langle A\phi U, X \rangle.$$ 

Hence, we obtain Assertion (i). Furthermore, with the help of Assertion (i), by considering the coefficient of $\phi U$ in (10), we obtain Assertion (j).

Finally, with the help of (9) and Assertion (d), after putting $Z = W = H$ in Lemma 2, we get

$$\langle A\phi H, Y \rangle \langle FH + c\phi H, X \rangle = \langle A\phi H, X \rangle \langle FH + c\phi H, Y \rangle$$

for any $X, Y \in \mathcal{D}_x$. In a similar manner as in the proof of assertions (a) and (b), we obtain assertions (k) and (l). \qed 

**Lemma 4** Let $M$ be a real hypersurface in a complex space form $M_n(c), n \geq 3$ with $\eta$-parallel shape operator, and let $x \in M_0$. Suppose $AU = \beta \xi + \lambda U$, for some $\lambda \in \mathbb{R}$. Then $A\phi U = \tilde{\lambda} \phi U$, for some $\tilde{\lambda} \in \mathbb{R}$.

**Proof** Suppose there exists a unit vector $H(\perp U)$ in $\mathcal{D}_x$ and $\tilde{\lambda}, \tilde{\delta} \in \mathbb{R}$ with $\tilde{\delta} \neq 0$ such that $A\phi U = \tilde{\lambda} \phi U + \tilde{\delta} \phi H$.

By making the following substitutions for the vectors $X, Y, Z$ and $W$ in Lemma 2:

- (a) $X = U, Y = Z = \phi U, W = H$;
- (b) $X = H, Y = Z = \phi H, W = U$;
- (c) $X = H, Y = \phi H, Z = U, W = \phi U$;
- (d) $X = U, Y = \phi U, Z = H, W = \phi H$;
- (e) $X = W = H, Y = Z = U$,
with the help of Lemma 3, we obtain the following equations:

\[ \tilde{\lambda}(F\phi U, H) = c\tilde{\delta} - \delta(FU, \phi U) \]  
\[ \tilde{\tau}(FU, \phi H) = c\delta - \delta(FH, \phi H) \]  
\[ -2c(\lambda - \tilde{\lambda}) = (\tau + \tilde{\tau})(FU, \phi U) + \delta(FHU, H) \]  
\[ -2c(\tau - \tilde{\tau}) = (\lambda + \tilde{\lambda})(FH, \phi H) + \delta(FU, \phi H) \]  
\[ c(\lambda - \tau) + \lambda^2 \tau - \lambda \tau^2 + \beta^2 \tau = 0. \]

It follows from (11), (12) and Lemma 3(b), (l) that

\[ \tilde{\lambda}\{(F\phi U, H) + (FU, \phi H)} = 2c\tilde{\delta} = \tilde{\tau}\{(F\phi U, H) + (FU, \phi H)} \]

which implies that

\[ \tilde{\tau} = \tilde{\lambda}. \]

Furthermore, by considering Lemma 3(e), (g) and (16), we see that

\[ 2\lambda \tau = \tilde{\lambda}(\lambda + \tau). \]

On the other hand, from Lemma 3(e), (11) and (17), we obtain

\[ \langle F\phi U, U \rangle + \langle F\phi H, H \rangle = 0. \]

Next, with the help of (17), Lemma 3(b), (e), (g), (l) imply these two equations:

\[ \tau \langle FU, \phi U \rangle = c(\tilde{\lambda} - \tau) \]
\[ \tilde{\lambda}(FH, \phi H) = c(\tilde{\lambda} - \lambda). \]

Finally, by summing up (13) and (14), and taking account of (16)–(21), we see that \( \tilde{\lambda}^2 - \tilde{\delta}^2 = \lambda \tau \), which, together with Lemma 3(j), implies that \( \lambda \tau = -c \). By applying this to (15), we have \( \beta^2 \tau = 0 \). This is a contradiction and the proof is completed.

**Theorem 8** Let \( M \) be a real hypersurface in a complex space form \( M_n(c), n \geq 3 \) with \( \eta \)-parallel shape operator. Suppose \( \beta \) never vanishes on \( M \) and \( AU = \beta \xi + \lambda U \), for some function \( \lambda \) on \( M \). Then \( M \) is locally congruent to a ruled real hypersurface.

**Proof** We have shown in Lemma 4, under these assumptions that \( U \) is an eigenvector for both \( \phi^2A\phi^2 \) and \( \phi A\phi \) at the points where \( \beta \neq 0 \). According to Theorem 7, \( M \) is locally congruent to a ruled real hypersurface, or a real hypersurface of type \( A \) or \( B \). As \( \beta \) never vanishes on \( M \), or equivalently, \( \xi \) is not principal in \( M \), \( M \) can never be of type \( A \) or \( B \). Hence, we conclude that \( M \) is locally congruent to a ruled real hypersurface.

**5 Real hypersurfaces with no common eigenvectors in \( D \) for both \( \phi^2A\phi^2 \) and \( \phi A\phi \)**

In Sect. 3, we showed that the existence of a common eigenvector in \( D_x \) for both \( \phi^2A\phi^2 \) and \( \phi A\phi \) is sufficient for the commutativity of \( \phi^2A\phi^2 \) and \( \phi A\phi \) at the point \( x \). In contrast, we shall now consider another possibility, i.e., \( \phi^2A\phi^2 \) and \( \phi A\phi \) do not share any eigenvector in \( D_x \), and show that this case indeed cannot occur on \( M_0 \).

Throughout this section, we suppose that \( x \in M_0 \) and that each vector in \( D_x \) fails to be an eigenvector for both \( \phi^2A\phi^2 \) and \( \phi A\phi \). Hence, for an arbitrary set \( \{\xi, E_1, \ldots, E_{2n-2}\} \) of
orthonormal eigenvectors for $\phi^2 A \phi^2$ at $x$, we may write, for $j \in \{1, \cdots, 2n - 2\}$,

$$AE_j = \beta_j \xi + \lambda_j E_j$$
$$A\phi E_j = \tilde{\beta}_j \xi + \tilde{\lambda}_j \phi E_j + \tilde{\delta}_j \phi H_j$$

where $H_j \in \mathcal{D}_x$ is a unit vector with $H_j \perp E_j$ and $\beta_j, \lambda_j, \tilde{\beta}_j, \tilde{\lambda}_j, \tilde{\delta}_j \in \mathbb{R}$ with $\tilde{\delta}_j \neq 0$.

Our main purpose here is to show under the above setting that $AU = \beta \xi + \lambda U$ at such a point $x$ in $M_0$. We begin with deriving two equations that are applicable later.

By letting $Y = Z = W = E_j$ in Lemma 2, we obtain

$$2\lambda_j (A^2 E_j, X) - 2\lambda_j (A^2 E_j, E_j) (E_j, X) = \langle FE_j, E_j \rangle (\lambda_j \phi E_j + 3A\phi E_j, X) \tag{22}$$

for any $X \in \mathcal{D}_x$. In addition, (22) can be simplified as

$$2\lambda_j \beta_j (\beta U - \beta_j E_j) = \langle FE_j, E_j \rangle (\lambda_j + 3\tilde{\lambda}_j) \phi E_j + 3\tilde{\delta}_j \phi H_j \tag{23}.$$ 

Lemma 5 \( \langle FE_j, E_j \rangle = 0 \), for $j \in \{1, \ldots, 2n - 2\}$.

Proof Suppose there is an $E_j$, say $E_1$, such that $\langle FE_1, E_1 \rangle \neq 0$. Then by considering the coefficient of $\phi H_1$ in (23) and taking account of $\tilde{\delta}_1 \neq 0$, we obtain

$$\lambda_1 \beta_1 \neq 0, \tag{24}$$

and so

$$\beta U = \frac{1}{2\lambda_1} E_1 + \frac{\langle FE_1, E_1 \rangle}{2\lambda_1 \beta_1} (\lambda_1 + 3\tilde{\lambda}_1) \phi E_1 + 3\tilde{\delta}_1 \phi H_1 \tag{25}.$$ 

Further, it follows from (22) and (24) that

$$A^2 E_1 \in \text{span}\{E_1, \phi E_1, \phi H_1, \xi\}. \tag{26}$$

Next, if we put $Y = H_1$ and $Z = W = E_1$ in Lemma 2 then

$$0 = \langle FE_1, E_1 \rangle \langle \phi AH_1 + A\phi H_1, X \rangle + 2 \langle FE_1, H_1 \rangle \langle A\phi E_1, X \rangle \tag{27}$$

for any $X \in \mathcal{D}_x$. Since $\tilde{\delta}_1 \langle FE_1, E_1 \rangle \neq 0$, after putting $X = \phi E_1$ in (27), we get

$$\tilde{\lambda}_1 \langle FE_1, H_1 \rangle \neq 0. \tag{28}$$

By substituting $Y = E_1$ and $Z = W = H_1$ in Lemma 2, we have

$$4c \langle AH_1, \phi H_1 \rangle \langle \phi E_1, X \rangle = \langle FH_1, H_1 \rangle (\lambda_1 \phi E_1 + A\phi E_1, X) + 2 \langle FE_1, H_1 \rangle \langle A\phi H_1, X \rangle \tag{29}$$

for any $X \in \mathcal{D}_x$. Since $\langle FE_1, H_1 \rangle \neq 0$, by putting $X = H_1$ in (29) and taking account of $H_1 \perp A\phi E_1, E_1$,

$$\langle A\phi H_1, H_1 \rangle = 0. \tag{30}$$

Furthermore, the following can also be deduced

$$\langle FH_1, H_1 \rangle (\lambda_1 + \tilde{\lambda}_1) = -2\tilde{\delta}_1 \langle FH_1, E_1 \rangle \neq 0, \tag{31}$$

$$A\phi H_1 = \tilde{h}_1 \xi + \tilde{\delta}_1 \phi E_1 + \tilde{\tau}_1 \phi H_1, \quad (\tilde{h}, \tilde{\tau} \in \mathbb{R}). \tag{32}$$

By virtue of (27) and (32), we can see that $AH_1 \in \text{span}\{E_1, H_1, \xi\}$. However, since $AH_1 \perp E_1$, we conclude that

$$AH_1 = h_1 \xi + \tau_1 H_1, \quad (h_1, \tau_1 \in \mathbb{R}). \tag{33}$$
Finally, by substituting $Y = Z = W = H_1$ in Lemma 2, with the help of (33), we get

$$2\tau_1 h_1 (\beta U - h_1 H_1) = \langle FH_1, H_1 \rangle \{(\tau_1 + 3\bar{\tau}_1)\phi H_1 + 3\bar{\delta}\phi E_1\}.$$ 

Since $\delta(FH_1, H_1) \neq 0$, we deduce that $\tau_1 h_1 \neq 0$ and hence

$$\beta U = \frac{1}{2\tau_1} H_1 + \frac{\langle FH_1, H_1 \rangle}{2\tau_1 h_1} \{(\tau_1 + 3\bar{\tau}_1)\phi H_1 + 3\bar{\delta}\phi E_1\}. \tag{34}$$

By comparing (25) and (34), as $H_1, E_1, \phi H_1, \phi E_1$ are orthogonal, $\lambda_1^{-1} = \tau_1^{-1} = 0$, which is a contradiction and this completes the proof.

**Lemma 6** $\lambda_j \neq 0$, for $j \in \{1, \ldots, 2n - 2\}$.

**Proof** Suppose to the contrary that $\lambda_j = 0$ for some $j$, say $\lambda_1 = 0$. Let $Y = E_1, Z = W = \phi E_1$ in Lemma 2. Then

$$2c\delta_1 (H_1, X) - 2\beta_1 \bar{\beta}_1 \langle A\phi E_1, X \rangle = \langle F\phi E_1, \phi E_1 \rangle \langle A\phi E_1, X \rangle$$

for any $X \in D_1$. Since $H_1 \perp A\phi E_1$, this equation gives $\delta_1 = 0$. This contradicts the hypothesis and the proof is completed.

**Lemma 7** Let $M$ be a real hypersurface in a complex space form $M_n(c), n \geq 3$, with $\eta$-parallel shape operator $A$. Then $\phi^2 A\phi^2$ and $A\phi$ share at least one eigenvector in $D_x$, for each $x \in M_0$.

**Proof** Let $x \in M_0$ such that $\phi^2 A\phi^2$ and $A\phi$ do not share any eigenvector in $D_x$. It follows from (23), Lemma 5 and Lemma 6 that

$$\beta_j (\beta U - \beta_j E_j) = 0, \quad j \in \{1, \ldots, 2n - 2\}.$$ 

Since $\xi$ is not principal, there is at least one (in fact, only one) $j$, say $j = 1$, such that $\beta_1 \neq 0$. Consequently, we obtain $U = \pm E_1$ is an eigenvector for $\phi^2 A\phi^2$. Furthermore, by Lemma 4, $U$ is an eigenvector for $A\phi$ too. This is a contradiction and the proof is completed.

6 Proof of Theorem 6

We are now in a position to classify real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator.

Since real hypersurfaces of type $A$ or $B$ and ruled real hypersurfaces have $\eta$-parallel shape operator, we only need to prove the sufficiency. By virtue of Theorem 3, we conclude that each component of the interior of $M \setminus M_0$, int$(M \setminus M_0)$, is of type $A$ or $B$. By verifying the constant principal curvatures for real hypersurfaces of type $A$ and $B$ (cf. the lists in [2, 17]), we can see that $|\phi A\phi|$ is locally a positive constant on int$(M \setminus M_0)$.

On the other hand, at each point $x \in M_0$, we have shown in Lemma 7 that there is at least one eigenvector in $D_x$ shared by both $\phi^2 A\phi^2$ and $A\phi$. Since $\xi$ is not principal at each point of $M_0$, the open submanifold $M_0$ can never be of type $A$ or $B$. According to Theorem 7, the open submanifold $M_0$ is locally congruent to a ruled real hypersurface and hence $|\phi A\phi| = 0$ on $M_0$. By a standard topological argument, either $M_0$ is empty or $M_0 = M$. This completes the proof.

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