An Approximation of Error Concentration Parameter for Simultaneous Circular Functional Relationship Model

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Abstract

In this study, we propose a simple procedure for obtaining estimate of the error concentration parameter of the von Mises distribution for the simultaneous circular functional relationship model. This procedure can be used to estimate the error concentration parameter for any value of ratio of error concentration parameter, \( \lambda_j \). Currently the error concentration parameter for this model can only be estimate by assuming the ratio of the error concentration parameter \( \lambda_j \) is equal to one. This new proposed procedure have been derived based on the expansion of the asymptotic power series of the modified Bessel function. Finally, the simulation study has been carried out to verify the accuracy of this new proposed procedure.

Keywords: Simultaneous circular functional model, Bessel function, ratio of error concentration parameter, von Mises distribution.
1. Introduction

The error-in-variables model differs from the ordinary or classical linear regression model in that the true independent variables or the explanatory variables are not observed directly, but are masked by measurements error. If $X$ is a mathematical variables, this termed as a linear functional relationship model and if $X$ is a random variables, then this is termed a linear structural relationship model between $X$ and $Y$.

In this paper, we consider the relationship when all variables involve are circular which takes values on the circumference of a circle, i.e. they are angles in the range $(0, 2\pi)$ radians or $(0^\circ, 360^\circ)$. The wind direction data measured by two different methods, which are the anchored wave buoy and HF radar system is one of the examples for this type of data and also known as the directional or circular data. Since we are considering the relationship between several circular variables, we refer the model as the simultaneous linear circular functional relationship model or fitting several circular functional relationship model. We also assume the errors of circular variables $X$ and $Y_j$ are independently distributed and follow the von Mises distribution with mean zero and concentration parameters $\kappa$ and $\nu_j$, respectively. For any circular random variable $\theta$, it is said to have a von Mises distribution if its probability distribution function is given by

$$g(\theta; \mu_0, \kappa) = \frac{1}{2\pi I_0(\kappa)} \exp\{\kappa \cos(\theta - \mu_0)\},$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero. The parameter $\mu_0$ is the mean direction while the parameter $\kappa$ is described as the concentration parameter.

We will propose the mathematical approach on how to find the estimation of error concentration parameters for the simultaneous linear circular functional relationship model by assuming the ratio of error concentration parameters, $\lambda_j = \frac{\nu_j}{\kappa}$ is known and only the unreplicated data available. Next, by using various approximation and asymptotic properties of Bessel function we can find the estimate of error concentration parameter for any value of ratio of error concentration parameters.

2. The Model

Suppose $x_i$ and $y_{ji}$ are observed values of the circular variables $X$ and $Y_j$ respectively, thus $0 \leq x_i, y_{ji} < 2\pi$, for $i = 1, \ldots, n$ and $j = 1, \ldots, q$. For any fixed $X_i$ and $Y_{ji}$, we assume that the observations $x_i$ and $y_{ji}$ (which are unreplicated) have been measured with errors $\delta_i$ and $\epsilon_{ji}$ respectively. Thus the full model can be written as $x_i = X_i + \delta_i$ and $y_{ji} = Y_{ji} + \epsilon_{ji}$, where $Y_{ji} = \alpha_j + \beta_j X_{i \mod 2\pi}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, q$.

We also assume $\delta_i$ and $\epsilon_{ji}$ are independently distributed with (potentially different) von Mises distributions, that is $\delta_i \sim VM(0, \kappa)$ and $\epsilon_{ji} \sim VM(0, \nu_j)$. Suppose we assume that the ratio of error concentration parameters, that is $\lambda_j = \frac{\nu_j}{\kappa}$ is known, then the log likelihood function is given by

$$\log l(\alpha_j, \beta_j, \kappa, X_1, \ldots, X_n; \lambda_j, x_1, \ldots, x_n, y_{j1}, \ldots, y_{qn}) =$$

$$-2n\log(2\pi) - n\log I_0(\kappa) - n\sum_{j=1}^{q} \log I_0(\lambda_j \kappa) + \sum_{i=1}^{n} \cos(x_i - \lambda_j \kappa) + \sum_{j=1}^{q} \sum_{i=1}^{n} \cos(y_{ji} - \alpha_j - \beta_j X_{i \mod 2\pi}).$$

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In the next section we will show on how to estimate the error concentration parameter \( \kappa \) when \( \lambda_j = 1 \). Further, by assuming \( \lambda_j = \lambda \), we will extend the case using the asymptotic properties of the Bessel function to estimate \( \kappa \) for any value of \( \lambda \).

### 3. Estimation of the Error Concentration Parameter \( \kappa \)

By setting \( \frac{\partial \log L}{\partial \kappa} = 0 \) of the log likelihood function we get the equation

\[
A(\kappa) + \sum_j \lambda_j A(\lambda_j, \kappa) = \frac{1}{n} \left( \sum_i \cos(x_i - X_j) + \sum_j \lambda_j \sum_i \cos(y_{i\beta} - \alpha_j - \beta_j X_j) \right),
\]

(1)

The approximation given by Dobson (1978), which is

\[
A^{-1}(w) = \frac{9 - 8w + 3w^2}{8(1 - w)}
\]

and by Best & Fisher (1997), which is

\[
\begin{align*}
&\lambda \geq 85.0, \\
&\lambda < 85.0, \\
&\lambda = 85.0, \\
&\lambda > 85.0,
\end{align*}
\]

(4)

Hence, \( \kappa \) in (1) can only be used for the case when \( \lambda_j = 1.0 \). By assuming \( \lambda_j = \lambda \), the above equation (1) can be simplified as below,

\[
A(\kappa) + q \lambda A(\lambda \kappa) = \frac{1}{n} \left( \sum_i \cos(x_i - X_j) + \lambda \sum_i \cos(y_{i\beta} - \alpha_j - \beta_j X_j) \right),
\]

(2)

In this section we show that by using the asymptotic properties of the Bessel function we can find an estimate of \( \kappa \) for any value of \( \lambda \). From the asymptotic power series for the Bessel functions \( I_0(r) \) and \( I_1(r) \) in Abramowitz and Stegun (1965), we have

\[
A(r) = \frac{I_1(r)}{I_0(r)} = 1 - \frac{1}{2r} - \frac{1}{8r^2} - \frac{1}{8r^3} + O(r^{-4}).
\]

(3)

Hence,

\[
A(\kappa) = \frac{I_1(\kappa)}{I_0(\kappa)} = 1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3} + O(\kappa^{-4})
\]

and

\[
q \lambda A(\lambda \kappa) = q \lambda \left\{ 1 - \frac{1}{2\lambda \kappa} - \frac{1}{8\lambda^2 \kappa^2} - \frac{1}{8\lambda^3 \kappa^3} + O(\kappa^{-4}) \right\}.
\]

Thus equation (2) may be simplified giving the expression approximately given by

\[
8(1 + q\lambda - c)\kappa^3 - 4(1 + q)\kappa^2 - \left( 1 + \frac{q}{\lambda} \right) \kappa - \left( 1 + \frac{q}{\lambda^2} \right) = 0
\]

(4)

where

\[
c = \frac{1}{n} \left( \sum_i \cos(x_i - X_j) + \lambda \sum_j \sum_i \cos(y_{i\beta} - \alpha_j - \beta_j X_j) \right).
\]

By using the following procedure, it can be shown that the above cubic equation in \( \kappa \), i.e. equation (4) has only one positive real root and two complex roots, where \( \hat{\kappa} \) as the positive real root.

The first step is to find a suitable substitution so that equation (4) can be transformed into the form of \( x^3 + fx + g = 0 \). This can be done by define \( D = \left( \frac{f}{3} \right)^3 + \left( \frac{g}{2} \right)^2 \) and according to Rades and Westergen (1998) the rules are,
(i) if $D > 0$, there exist one real and two complex roots,
(ii) if $D = 0$, there exist three real roots (at least two are equal), and
(iii) if $D < 0$, there exist three distinct real roots.

The next step is to show that $D > 0$ in which there is only one positive real root for equation (4) and suppose $r_1, r_2,$ and $r_3$ be the roots, then

$$r_1 = u + v,$$
$$r_2 = -\left(\frac{u + v}{2}\right) + \left(\frac{u - v}{2}\right)\sqrt{3}i,$$ and

$$r_3 = -\left(\frac{u + v}{2}\right) - \left(\frac{u - v}{2}\right)\sqrt{3}i.$$ where

$$u = \left(-\frac{g}{2} + \sqrt{D}\right)^{\frac{1}{3}} \text{ and } v = \left(-\frac{g}{2} - \sqrt{D}\right)^{\frac{1}{3}}.$$ Our aim is to solve the following equation for $\kappa$, which is

$$8(1 + q\lambda - c)\kappa^3 - 4(1 + q)\kappa^2 - \left(1 + \frac{q}{\lambda}\right)\kappa - \left(1 + \frac{q}{\lambda^2}\right) = 0$$
or

$$a_0\kappa^3 + a_1\kappa^2 + a_2\kappa + a_3 = 0,$$ where

$$a_0 = 8(1 + q\lambda - c) > 0, \text{ since } \lambda > 0 \text{ and }$$
$$c = \frac{1}{n}\left\{\sum_i \cos(x_i - X_i) + \lambda\sum_j \sum_i \cos(y_{ji} - \alpha_j - \beta_i, X_i)\right\} < (\lambda + 1)$$
$$a_1 = -4(1 + q), \text{ } a_2 = -\left(1 + \frac{q}{\lambda}\right) \text{ and } a_3 = -\left(1 + \frac{q}{\lambda^2}\right).$$

A suitable substitution is $\kappa = y - \left(\frac{a_1}{3a_0}\right)$ in which equation (5) may be simplified to

$$y^3 + fy + g = 0,$$ where

$$f = \frac{3a_0a_2 - a_1^2}{3a_0^2} \text{ and } g = \frac{2a_1^3 - 9a_0a_2a_0 + 27a_3a_0^2}{27a_0^3}.$$ Further, the coefficients of $f$ and $g$ may be written as

$$f = -\left(\frac{3\Delta(\Delta + q) + 2\lambda(1 + q)^2}{24\lambda\Delta^2}\right) \text{ and } g = -\left(\frac{16\lambda^2 + 9\lambda(\lambda + 1)\Delta + 27(\lambda^2 + 1)\Delta^2}{216\lambda^2\Delta^3}\right)$$

where $\Delta = 1 + q\lambda - c$.

Next, in the following step we will show that $D = \left(\frac{f}{3}\right)^3 + \left(\frac{g}{2}\right)^2 > 0$.

This can be done by direct substitution for $f$ and $g$, and it can be shown that

$$D = \frac{3(3\Delta)^2}{(864\lambda^2\Delta^3)^2}\left\{\left(1 + q\right)^2\lambda^2\left[\lambda - q\right]^2 + 14\left(\lambda^2 + q^2\right) + 16q\left(\lambda^2 + 1\right)\right] + 2\Delta\left(q + \lambda\right)\lambda\left[\lambda - q\right]^2 + 16\left(\lambda^2 + q^2\right) + 18q\left(\lambda^2 + 1\right)\right\} + 108\Delta^2\left(\lambda^2 + q^2\right)^2$$

which is always positive for all $\lambda$ and $\Delta$. 
Hence $D > 0$ was satisfied which suggest that equation (4) has only one real root given by $(u + v)$ and two complex roots. Our final step is to show that this real root is always positive by showing

$$u + v = \left(\frac{-g}{2} + \sqrt{D}\right)^{\frac{1}{3}} + \left(\frac{-g}{2} - \sqrt{D}\right)^{\frac{1}{3}} > 0.$$ 

Let $u = \left(\frac{-g}{2} + \sqrt{D}\right)^{\frac{1}{3}}$ and $v = \left(\frac{-g}{2} - \sqrt{D}\right)^{\frac{1}{3}}$. If $(u + v) > 0$, then it is very obvious that $(u + v)^{3} > 0$. Further

$$(u + v)^{3} = u^{3} + v^{3} + 3uv(u + v) = -g - f(u + v)$$

which is always positive since $f, g < 0$ and $u > v$.

Therefore, we conclude that the equation

$$8(1 + q\lambda - c)\kappa^{3} - 4(1 + q)\kappa^{2} - \left(1 + \frac{q}{\lambda}\right)\kappa - \left(1 + \frac{q}{\lambda^{2}}\right) = 0$$

has only one positive real root which is given by

$$\hat{\kappa} = \left(\frac{-g}{2} + \sqrt{D}\right)^{\frac{1}{3}} + \left(\frac{-g}{2} - \sqrt{D}\right)^{\frac{1}{3}} - \left(\frac{a_{1}}{3a_{0}}\right).$$

4. Simulation results

Simulation study have been carried out to verify the accuracy of error concentration parameter $\kappa$ obtained by using the above proposed procedure. For different value of $\lambda$ and $\kappa$, let $s$ be the number of simulations and the following computation were carried out from the simulation study.

i) Mean, $\bar{\kappa} = \frac{1}{s} \sum \hat{\kappa}_{j}$,

ii) Estimated Bias $= \bar{\kappa} - \kappa$

iii) Absolute Relative Estimated Bias (%) $= \left(\frac{\bar{\kappa} - \kappa}{\kappa}\right) \times 100\%$,

iv) Estimated Standard Errors $= \sqrt{\frac{1}{s-1} \sum (\hat{\kappa}_{j} - \bar{\kappa})^{2}}$,

v) Estimated Root Mean Square Errors (RMSE) $= \sqrt{\frac{1}{s} \sum (\hat{\kappa}_{j} - \kappa)^{2}}$.

The simulation results with $s = 5000$ and sample size of 100 for each set of true parameter value of $\kappa$ and $\lambda$ are shown in Table 1.

It is appears that the estimate bias and the absolute relative bias are very small as well as the estimate standard error and estimate root mean square error for all value of parameters. This suggests that the estimate obtained by using this technique is very close the true parameter value.
Table 1: Simulation results for different value of $\lambda$ and true value of $\kappa$.

<table>
<thead>
<tr>
<th></th>
<th>$\lambda = 1, \kappa = 1$</th>
<th>$\lambda = 1.5, \kappa = 2$</th>
<th>$\lambda = 2, \kappa = 2$</th>
<th>$\lambda = 2, \kappa = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean of $\kappa$</td>
<td>1.23827</td>
<td>1.97122</td>
<td>1.98760</td>
<td>2.98713</td>
</tr>
<tr>
<td>Est Bias</td>
<td>0.23827</td>
<td>-0.02878</td>
<td>-0.01240</td>
<td>-0.01287</td>
</tr>
<tr>
<td>Abs Rel Est Bias (%)</td>
<td>0.23827</td>
<td>0.01439</td>
<td>0.00620</td>
<td>0.00429</td>
</tr>
<tr>
<td>Est S.E</td>
<td>0.056269</td>
<td>0.13884</td>
<td>0.14285</td>
<td>0.22555</td>
</tr>
<tr>
<td>Est RMSE</td>
<td>0.24483</td>
<td>0.14178</td>
<td>0.14337</td>
<td>0.22589</td>
</tr>
</tbody>
</table>

5. Conclusion
This study has shown that by using the proposed technique we can find the estimate of error concentration parameter, $\kappa$ for any value of $\lambda$ as opposed to the approximation given by Dobson (1978) and Best & Fisher (1979) which can only be used when the ratio of error concentration parameter is equal to one or equal error concentration parameter. The simulation study also suggest that the estimation is more accurate when $\kappa > 1$.

References